5.3 Integration

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.3 Integration
Difficulty	Very Hard

Time allowed: 110

Score: /91

Percentage: /100

Question 1a

A function f is a piecewise linear function defined by

$$f(x) = \begin{cases} \dfrac{1}{2}(x+7), & x \leq 3 \\[2mm] 5, & 3 < x < 10 \\[2mm] 35 - 3x, & x \geq 10 \end{cases}$$

(a) Sketch the graph of $y = f(x)$ in the interval $0 \leq x \leq 12$.

[3 marks]

Question 1b

(b) Use your sketch from part (a), along with relevant area formulae, to work out the value of the integral

$$\int_{1}^{11} f(x)\, \mathrm{d}x$$

You should *not* use your GDC to find the value of the integral.

[4 marks]

Question 2

The derivative of the function f is given by

$$f'(x) = -\frac{3}{x^2} + \frac{1}{2}x^2 - \frac{2}{3}x + 2, \quad x > 0$$
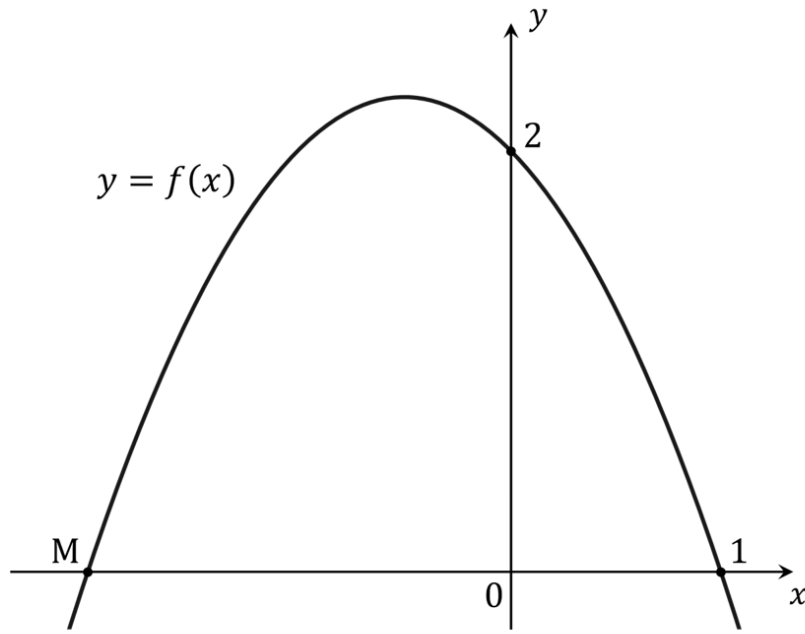
and the curve $y = f(x)$ passes through the point $\left(6, \frac{65}{2}\right)$.

Find an expression for f.

[7 marks]

Question 3a

A curve $y = f(x)$ has the gradient function $f'(x) = ax - 1$. The diagram below shows part of the curve, with the x- and y-intercepts labelled.

(a) Find

    (i)   the value of a

    (ii)   the equation of the curve $y = f(x)$

    (iii) the coordinates of point M.

[5 marks]

Question 3b

(b) Find the area of the region enclosed by the curve and the x-axis.

[3 marks]

Question 4

Celebrity chef Pepper Bee has opened a new restaurant and is charging diners £630 for a piece of his signature 'Croesus' cake. The chef claims that the price reflects the high cost of the gold foil that is placed on top of each slice of cake, but a suspicious and disgruntled customer has decided to investigate this claim.

The shaded area in the diagram below shows the shape of the piece of gold foil that is placed on top of each slice of cake:
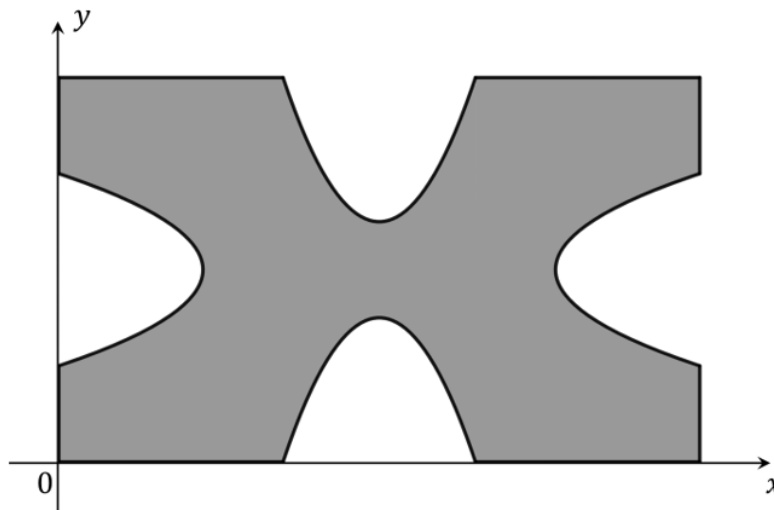
The shape is that of a rectangle, from which four identical curved sections have been removed. The rectangle is bounded by the positive x- and y-axes and the lines $x = 10$ and $y = 6$. The shape of one of the curved sections in the diagram can be described by the curve with equation

$$y = -\frac{1}{4}(2x - 7)(2x - 13)$$

All units are given in centimetres.

Given that gold foil costs £ 0.004788 per mm^2, work out the cost of the gold foil on a piece of Pepper Bee's Croesus cake. Give your answer to 2 decimal places.

[6 marks]

Question 5

A company is designing a piece for one of the plastic wargaming models they produce. The piece is to be in the form of a prism, with a cross-sectional area as indicated by the shaded region R in the following diagram:
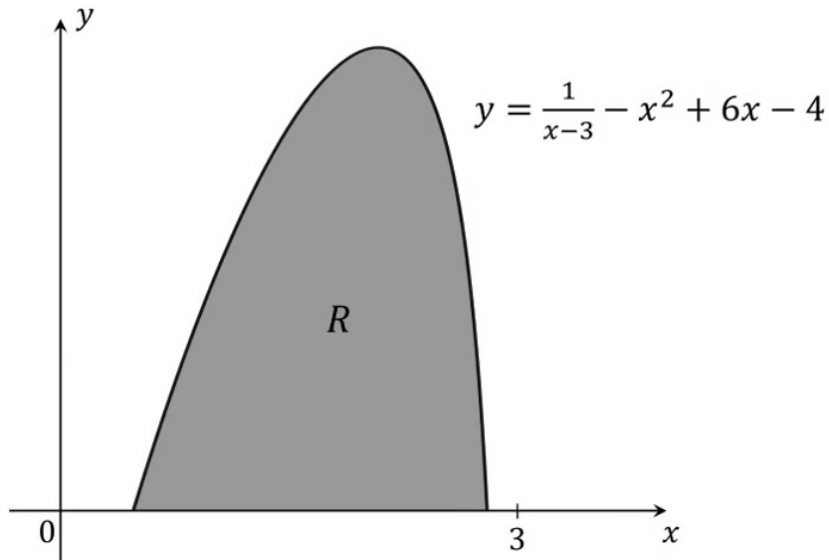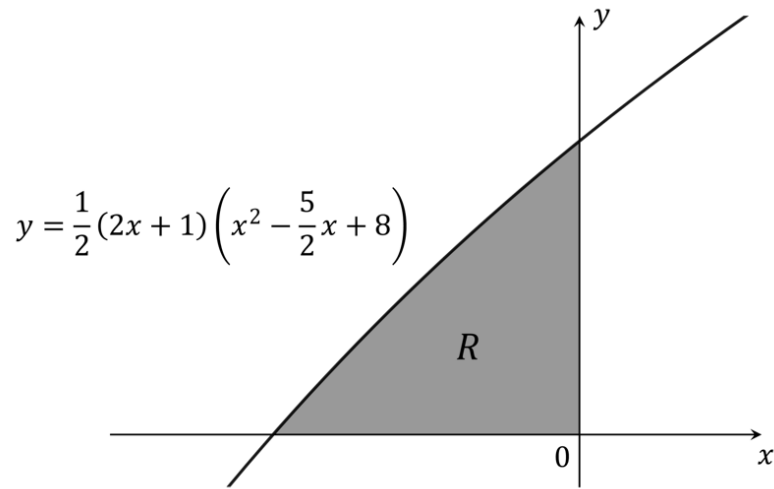

$$y = \frac{1}{x-3} - x^2 + 6x - 4$$

Region R is bounded, as shown, by the positive x-axis and the curve with equation $y = $. All units are in centimetres.

Given that the model piece will have a volume of 50.3 cm^3, find the length of the piece.

[6 marks]

Question 6a

The following diagram shows part of the graph of $f(x) = \frac{1}{2}(2x + 1)\left(x^2 - \frac{5}{2}x + 8\right), x \in \mathbb{R}$. The shaded region R is bounded by the x-axis, the y-axis and the graph of f.
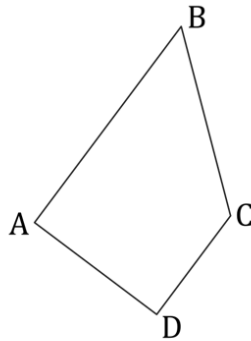

(a) Find the area of region R.

[3 marks]

Question 6b

A trapezoid ABCD is shown below.

[AB] is perpendicular to [AD] and parallel to [CD]. $CD = \dfrac{34}{45}$. The coordinates of points A, B and D are $(0, 1)$, $\left(\dfrac{9}{10}, p\right)$ and $\left(\dfrac{3}{4}, \dfrac{7}{16}\right)$ respectively, where $p > 0$ is a constant.

(b) Given that ABCD has the same area as the region R above, find the value of p, the y-coordinate of point B.

[6 marks]

Question 7a

A curve has the equation $y = x^3 - 5x^2 + 2x + 8$.

(a) Sketch the curve.

[2 marks]

Question 7b

(b) Using the trapezoidal rule with $n = 5$, determine an approximation for the integral

$$\int_{-1}^{\frac{3}{2}} (x^3 - 5x^2 + 2x + 8)\, \mathrm{d}x$$

Give your answer as an exact value.

[5 marks]

Question 7c

(c) Explain, using your sketch from part (a), why your approximation will be an underestimate.

[3 marks]

Question 7d

(d) (i) Find the exact value of the integral from part (b).

(ii) Find the percentage error of the approximation found in part (b), compared with the exact value of the integral.
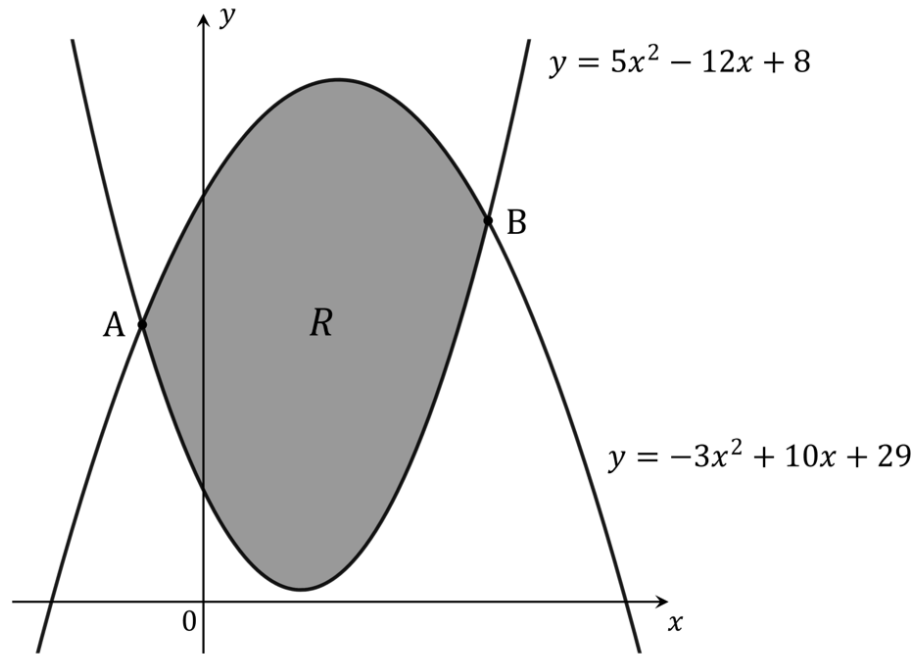
[3 marks]

Question 7e

(e) Explain how you might modify your method in part (b) in order to get a more accurate approximation.

[1 mark]

Question 8a

The shaded region R in the following diagram is bounded by the two curves $y = 5x^2 - 12x + 8$ and $y = -3x^2 + 10x + 29$.

The two curves intersect at points A and B as shown. x_A and x_B are the x-coordinates of points A and B respectively.

(a) By setting up and solving an appropriate quadratic equation, find the values of x_A and x_B.

[4 marks]

Question 8b

(b) Find the area of region R, giving your answer as an exact value.

[6 marks]

Question 9a

For a particle P travelling in a straight line, the velocity, v m/s, of the particle at time t seconds is given by the equation

$$v(t) = t^3 - 15t^2 + 48t + 64, \qquad 0 \leq t \leq 10$$

At time t_1 the particle reaches its maximum velocity, while at time t_2 the particle comes momentarily to rest.

(a) Find the values of t_1 and t_2, justifying your answers in each case.

[4 marks]

Question 9b

The distance travelled between two times by a particle moving in a straight line may be found by finding the area beneath the particle's velocity-time graph between those two times.

(b) Find

(i) the total distance travelled by the particle P between times $t = 0$ and $t = 10$.

(ii) the percentage of that total distance that is covered between times t_1 and t_2.

[5 marks]

Question 10a

Donty is a would-be social media celebrity who is obsessed with the number of 'likes' his posts receive. He hires a statistician to study his social media accounts, and after analysing several years of data she determines that the rate of change of his number of 'likes' can be modelled by the equation

$$\frac{\mathrm{d}L}{\mathrm{d}x} = -0.164x^3 + 2.73x^2 - 12.7x + 15.3, \quad 0 \le x \le 12$$

where L represents the number of likes received on a given day (in thousands of likes), and x represents the amount of new video content Donty uploaded on the preceding day (in hours). Because of technical limitations, Donty is unable to upload more than 12 hours of new video content on any given day.

It is known as well that 36075 'likes' are received on a day after 5 hours of video content was uploaded the day before.

(a) Find the maximum and minimum number of 'likes' that Donty can expect to receive in a day, and the corresponding number of hours of new video content that Donty should upload on the preceding day to attain that maximum or minimum. Be sure to justify that the values you find are indeed the maximum and minimum possible.

[11 marks]

Question 10b

(b) (i) For the maximum value determined in part (a), calculate the number of likes that are received for each minute of new video content uploaded the preceding day.

(ii) State, with a reason, whether the value calculated in part (b) (i) represents the maximum number of 'likes per minute of new content' that Donty is able to achieve.

[4 marks]