3.9 Modelling with Vectors

Question Paper

Course	DP IB Maths
Section	3. Geometry & Trigonometry
Topic	3.9 Modelling with Vectors
Difficulty	Hard

Time allowed: 90

Score: /71

Percentage: /100

Question 1a

Two spaceships A and B, in a 3D virtual reality game, are moving such that their positions relative to a fixed point O at time t seconds, $0 \leq t < 30$, are defined by the position vectors $r_A = \begin{pmatrix} 2 \\ -3.5 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1.2 \\ 0.5 \\ 2 \end{pmatrix}$ and $r_B = \begin{pmatrix} -2 \\ 4 \\ 9.5 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 0.3 \end{pmatrix}$ respectively.

(a)
Show that the two spaceships are on course to collide at point P and write down the coordinates of P.

[4 marks]

Question 1b

Spaceship B reduces its velocity such that its position vector is now given by

$$r_B = \begin{pmatrix} -2 \\ 4 \\ 9.5 \end{pmatrix} + t\begin{pmatrix} 1.6 \\ -0.8 \\ 0.24 \end{pmatrix}$$

(b)
Show that spaceship B is still travelling in its original direction.

[1 mark]

Question 1c

(c)
Show that the distance between the two spaceships can be written as

$$\sqrt{4.9476t^2 - 56.62t + 144.5}$$

[5 marks]

Question 1d

(d)
Hence find the distance between the two spaceships when spaceship A is at P.

[2 marks]

Question 2a

A car is moving with constant velocity along the line with equation $r_c = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ 12 \end{pmatrix}$. A bird is perched at the point $(25,\ 32,8)$ and at $t = 0$, starts to fly at a constant velocity in the direction of the vector $(2\mathrm{i} + 31\mathrm{j} - 4k)$.

All distances are measured in metres and time in seconds. The base vectors i and j represent due east and due west respectively and the base vector $\boldsymbol{k}$ points upwards.

(a)
Verify that the bird does not collide with the car.

[2 marks]

Question 2b

(b)
Show that at some point in time the bird will be directly above the car and state the time at which this occurs.

[3 marks]

Question 2c

(c)
Hence find the distance between the bird and the car at that time.

[2 marks]

Question 3a

A hawk is flying with constant velocity, $\boldsymbol{v}$, measured in kilometres per minute, where

$$\boldsymbol{v} = \begin{pmatrix} -1 \\ a \\ 0.2 \end{pmatrix}.$$

A fixed public telescope is located at a point $A(1, -2)$ relative to an origin O. At time $t = 0$ minutes the hawk is at the point $(4, -1, 1)$ and at time $t = T$ minutes the hawk is vertically above the telescope.

The $\boldsymbol{x}$ direction is due east, the $\boldsymbol{y}$ direction is due north and the $\boldsymbol{z}$ direction is vertically upwards. All distances are measured in kilometres.

a)
Find the value of $\boldsymbol{a}$ and the height of the hawk when it is directly above the telescope.

[4 marks]

Question 3b

b)
Given that the hawk continues flying in the same direction, find
i)
the speed at which the hawk is flying,
ii)
the time at which the hawk is exactly 4 km from the telescope.

[6 marks]

Question 4a

An eagle is tracking a mouse on the ground that is moving with constant velocity such that its position vector relative to an origin O, at time t seconds can be modelled by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 30 \\ -10 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

At time $t = 3$ seconds the eagle is positioned at a point exactly 28 metres vertically above the mouse.

The x, y and z directions are due east, due north and vertically upwards respectively with all distances in metres.

a)
Write down a vector equation of a line the eagle should fly in if it is to take off at the time $t = 3$ seconds and reach the ground at the same place as the mouse.

[3 marks]

Question 4b

b)
During their subsequent motion, find the speed of
i)
the eagle,
ii)
the mouse.

[2 marks]

Question 4c

Luckily for the mouse, at the point where the eagle is exactly 5 metres from it, it finds a hole to hide in.

c)
Find the amount of time the mouse had left before the eagle would have reached it.

[3 marks]

Question 5a

A person is walking in a straight line and at a constant speed. The person moves from their initial point $(5,1)$, through a second point $(2,-6)$ and continues walking in a straight line. At the same time a leaf located at $(1,-5,14)$ falls from its starting position in a tree and moves with constant velocity in the direction of the vector $a\mathrm{i}+(a-2)\mathrm{j}-4.92\mathrm{k}$.

The x, y and z directions are due east, due north and vertically upwards respectively with all distances in metres.

a)
Given that the person is 1.7 metres tall and that they are walking on a horizontal plane, find the coordinates of the point at which the leaf lands on the top of their head.

[5 marks]

Question 5b

b)
The leaf takes 14.2 seconds to fall from its starting point in the tree to the point where it lands on the person's head. Find the speed with which it falls.

[3 marks]

Question 6a

The velocity of a tractor is given by $v = \begin{pmatrix} 10\sin t \\ -10\cos t \end{pmatrix}$. At the time, t seconds, the tractor is moving relative to an origin, O. At $t = 0$ the position vector of the tractor is $\begin{pmatrix} -10 \\ 0 \end{pmatrix}$.

a)
Find
i)
the initial speed of the tractor
ii)
a position vector of the tractor at time, t, relative to the origin.

[4 marks]

Question 6b

b)
Find the distance of the tractor from the origin at time, t.

[2 marks]

Question 6c

c)
Describe the shape of the path that the tractor is making.

[2 marks]

Question 7a

A ball is thrown from a height of 1.2 metres with initial velocity $\boldsymbol{u} = \begin{pmatrix} 4 \\ 15 \end{pmatrix} \text{ms}^{-1}$. The ball moves freely under gravity. The velocity of the subsequent motion of the ball can be modelled by the vector

$$\boldsymbol{v}(t) = \begin{pmatrix} u_x \\ u_y - gt \end{pmatrix}$$

where u_x is the horizontal component of the initial velocity and u_y is the vertical component of the initial velocity.

The ground below the point from which the ball was thrown can be considered the origin. It is assumed that any effects of air resistance will be negligible, the ball is modelled as a particle and $g = 9.81 \text{ ms}^{-2}$.

a)
Find
(i)
the displacement vector of the ball at the time, t,
(ii)
the maximum height of the ball,
(iii)
the angle of elevation of the ball as it was thrown.

[6 marks]

Question 7b

b)
Assuming that the ground is horizontal, find how far from the origin the ball is when it lands.

[3 marks]

Question 8a

A whale is moving through the sea such that its position, at time t seconds, is given by the vector

$$r = \begin{pmatrix} \frac{1}{3}t^2 - \frac{1}{2}t \\ \frac{1}{2}t^2 + 4t \end{pmatrix}$$

a)
Find
(i)
the initial speed of the whale,
(ii)
the speed of the whale after 10 seconds,
(iii)
the acceleration vector of the whale

[5 marks]

Question 8b

A turtle is riding on the back of the whale and at the time $t = 10$ seconds begins to swim away from it with velocity vector $v = \frac{1}{2}T^2\mathrm{i} + \frac{3}{4}T^2\mathrm{j}$, where $T = t - 10$ seconds. The whale continues to swim along the same path.

b)
Find the distance between the whale and the turtle 10 seconds after they separated.

[4 marks]