5.8 Advanced Differentiation

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.8 Advanced Differentiation
Difficulty	Hard

Time allowed: 110

Score: /90

Percentage: /100

Question 1

Let $f(x) = 2x^3$.

By differentiating from first principles, show that $f'(x) = 6x^2$.

[4 marks]

Question 2a

Let $f(x) = \cos\sqrt{2x}$.

(a)

Find the positive solution to the equation $f'(x) = f'''(x) - \frac{3\sqrt{2}}{2}$ that is closest to zero.

[4 marks]

Question 2b

(b)

(i)

Show that $f^{(4)}(x) = kf(x)$, where k is a constant to be determined.

(ii)

Write down the value of $f^{(10)}(0)$.

[3 marks]

Question 3a

(a)

Given that $f(x) = \arctan\left(\frac{1}{x}\right)$, find $f'(x)$.

[3 marks]

Question 3b

For the function g defined by $g(x) = 4^{-x} + 2\log_4 x$, show that

$$g'(x) = \frac{1}{(\ln 2)x} - (\ln 2)2^{-2x+1}$$

[4 marks]

Question 3c

(c)
Find the derivative of the function $h(x) = \operatorname{cosec}(3x^2)$.

[3 marks]

Question 4a

For the curve defined by $y = \cot x^2$, show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 8x^2(y^3 + y) - 2y^2 - 2$$

[6 marks]

Question 4b

(b)

For the curve defined by $y = \arcsin x$, show that

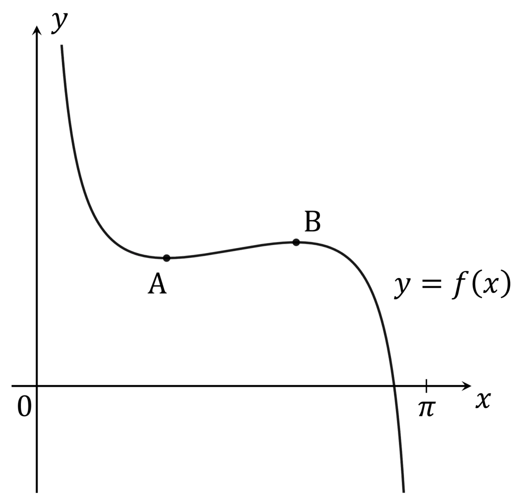
$$y'' = \frac{x}{(1-x^2)\sqrt{1-x^2}}$$

[4 marks]

Question 5a

Consider the function f defined by $f(x) = \tan\left(\frac{\pi}{2} - x\right) + \frac{4}{3}x,\ 0 < x < \pi.$

The following diagram shows the graph of the curve $y = f(x)$:



The points marked A and B are the turning points of the graph.

(a)
Find the coordinates of points A and B.

[6 marks]

Question 5b

(b)

Find the equation of the normal to the graph at the point where the x-coordinate is equal to $\frac{\pi}{4}$.

[4 marks]

Question 6a

For each of the following, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by differentiating implicitly with respect to x.

(a)

$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

[2 marks]

Question 6b

(b)

$$e^{x^2} + x = y^3 - y$$

[2 marks]

Question 6c

(c)

$$\frac{(x-y)^7}{x^2}=1$$

[3 marks]

Question 7a

Consider the curve defined by the equation

$$\frac{1}{\sqrt[3]{x^3-y^3}}=\frac{2\pi}{3}$$

(a)

Use implicit differentiation to find $\frac{dx}{dy}$ in terms of x and y.

[4 marks]

Question 7b

(b)

By first rewriting the equation of the curve in the form $y = f(x)$:

(i)

Determine the coordinates of the point on the curve where $\frac{dy}{dx} = 0$.

(ii)

Explain why $f(x)$ is an increasing function on all intervals $[a, b]$ for which the interval $]a, b[$ does not include the x-coordinate of the point identified in part (b)(i).

(iii)

Describe the asymptotic behaviour of the curve as $x \to \pm\infty$.

[7 marks]

Question 8a

After setting up a firework rocket on a stretch of level ground, the firework engineer lights the fuse and steps back to a safe distance of 10 metres from the rocket. The rocket then begins to ascend vertically into the air at a constant velocity of 64 metres per second.

Let D be the distance, in metres, between the rocket and the point on the ground where the engineer is standing at time t seconds after the rocket takes off. Let h be the height, in metres, of the rocket above the ground at time t seconds.

(a)
Write an expression for D in terms of h only.

[2 marks]

Question 8b

(b)
Use implicit differentiation to show that

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{64h}{\sqrt{h^2 + 100}}$$

[5 marks]

Question 8c

(c)

Find

(i)

the rate at which the distance between the rocket and the point where the engineer is standing is increasing 1.56 seconds after the rocket takes off.

(ii)

the height of the rocket above the ground at the moment when the distance between the rocket and the point where the engineer is standing is increasing at a rate of $10\ \mathrm{ms}^{-1}$.

[4 marks]

Question 8d

(d)

(i)

Describe the mathematical behaviour of $\frac{\mathrm{d}D}{\mathrm{d}t}$ as h becomes large and interpret this in the context of the question.

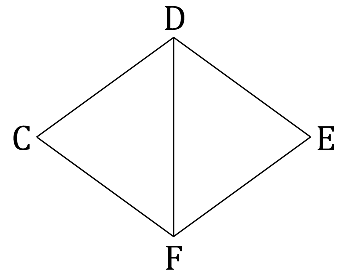
(ii)

Comment on the validity of the model for large values of h.

[4 marks]

Question 9a

Quadrilateral CDEF represents a corral for unicorns. There are fences along the four sides of the corral, as well as a straight fence across the middle connecting points D and F. Because of the way unicorns are trained, it is essential that triangles CDF and DEF be identical isosceles triangles, with $\mathrm{CD} = \mathrm{CF} = \mathrm{DE} = \mathrm{EF}$. The length of side DF, however, can vary.



Gonzolph is a unicorn trainer who is concerned about the high cost of unicorn fencing. He would therefore like the total length of fencing, P, used in his corral to be the minimum possible for a given area, A, to be enclosed.

Let $\mathrm{DF} = 2x$ m and let $\mathrm{CD} = \mathrm{y}$ m.

(a)

By first finding the derivative $\frac{\mathrm{d}A}{\mathrm{d}x}$ in terms of x and y, show that for a given area the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x^2 - y^2}{xy}$ must be satisfied.

[7 marks]

Question 9b

(b)

By considering the derivative $\frac{\mathrm{d}P}{\mathrm{d}x}$, show that when the length of fencing required to enclose a given area is the minimum possible then $x = \left(\frac{\sqrt{33}-1}{8}\right)y$.

[6 marks]

Question 9c

(c)

Hence find the size of angle $\mathrm{F\widehat{C}D}$ in a corral that minimises the amount of fencing required to enclose a given area.

[3 marks]