

1.6 Further Complex Numbers

Question Paper

Course	DPIB Maths
Section	1. Number & Algebra
Topic	1.6 Further Complex Numbers
Difficulty	Very Hard

Time allowed: 100
Score: /79
Percentage: /100

Question 1a

Let $z = 2 \operatorname{cis} \frac{3\pi}{4}$.

- a)
i)
Find the values of z^2 , z^3 and z^4 , giving your answers in the form $ae^{i\theta}$, where $a \in \mathbb{R}^+$ and θ is given as an exact value.
- ii)
Plot z , z^2 , z^3 and z^4 on an Argand diagram.

[5 marks]**Question 1b**

- b)
Find the exact value of a such that successive integer powers of the complex number $w = \frac{z}{a+i}$ lie on a unit circle.

[3 marks]

Question 2

Consider the complex numbers z and w , where $z = \sqrt{3} - i$, $\operatorname{Im}\left(\frac{z^2}{w}\right) = 0$, $\left|\frac{z^2}{w}\right| = \frac{1}{2}|z|$.

Use geometrical reasoning to find the two possibilities for w , giving your answers in exponential form.

[4 marks]

Question 3a

Consider the complex numbers $z_1 = 2e^{\frac{\pi}{3}i}$, $z_2 = 3 - z_1$ and $z_3 = \frac{z_1}{z_2}$.

a)

Write z_1 , z_2 and z_3 in the form $a + bi$ where $a, b \in \mathbb{R}$, giving exact values of a and b .

[4 marks]

Question 3b

b)

A scale model of a triangular patio is planned by representing the vertices of the triangle on an Argand diagram as the points z_1 , z_2 and z_3 . Find the area of the triangle in the model.

[5 marks]

Question 4a

Consider the complex number $z = 1 + \sqrt{3}i$.

a)

i)

Plot the position of z on an Argand diagram.

ii)

Express z in the form $z = ae^{ib}$, where $a, b \in \mathbb{R}$, giving the exact value of a and the exact value of b .

[3 marks]

Question 4b

b)

Use technology to find the value of $\sqrt{z^3}$. Give your answer in the form $z = ae^{ib}$, where $a, b \in \mathbb{R}$, giving the exact value of a and the exact value of b .

[2 marks]**Question 4c**

c)

Find the smallest positive integer k such that z^k is

i)

a positive real number,

ii)

a negative real number.

[2 marks]**Question 4d**

d)

Explain why there is no possible integer value of k such that z^k is purely imaginary.

[1 mark]**Question 5a**

Given the points 1 and z on an Argand diagram, where $z \neq 0$ is a complex number, explain how to find each of the following points by geometrical construction. In each case provide a sketch to illustrate your answer.

a)

z^2 .

[3 marks]

Question 5b

b)
 $(2 - i)z$.

[3 marks]

Question 6

Consider the equations $u^* + 2v = 2i$ and $iu + v^* = 3$, where $u, v \in \mathbb{C}$. Find $\frac{u}{v}$ giving your answer in the form $re^{i\theta}$, where $r > 0$ and $0 < \theta < 2\pi$.

[8 marks]

Question 7

By first expressing $1 + \sqrt{3}i$ and $-1 + i$ in the form $r \operatorname{cis} \theta$ where $r > 0$ and $-\pi < \theta \leq \pi$, show that $\frac{5\pi}{12} = 2 + \sqrt{3}$.

[8 marks]

Question 8a

Two power sources are connected to a single electrical circuit. At time t seconds, the voltage, V_1 , provided by the first power source is modelled by $V_1 = \text{Im}(3e^{12ti})$, and the voltage, V_2 , from the second power source can be modelled by the function V_2 .

The total voltage in the circuit, $V_T = V_1 + V_2$, is given by $V_T = 10 \sin(12t + 20)$.

a)

Find an expression for V_2 in the form $A \sin(Bt + C)$, where $A, B, C \in \mathbb{R}$.

[4 marks]

Question 8b

b)

Write down the maximum voltage and the phase shift provided by the second power source, giving your answers correct to 2 decimal places.

[2 marks]

Question 9a

A popular fast-food chain is considering opening a new restaurant in an up and coming part of New York. They have looked at competition in the area and predict that the costs C , in thousands of USD, to run the new restaurant for the first 100 days after opening could be modelled by the function

$$C(d) = 30 \sin(0.01d - 4.5) + 4, \quad 0 < d \leq 100,$$

where d is the number of days since opening.

The CEO of the company will only give permission for the new restaurant to open if the model predicts that the revenue will be greater than the costs by the 80th day after opening.

a)

The revenue R , in thousands of USD, for the first 100 days is predicted to follow the model

$$R(d) = a \sin(0.01d + 0.1) + 4.5, \quad 0 < d \leq 100, \quad a \in \mathbb{N}.$$

i)

Find the minimum value of a that will ensure the restaurant is making a profit by day 80.

ii)

For this value of a , show that the profits of the hotel can be modelled by the function $P(d) = A \sin(0.01d + b) + c$, giving the values of A , b and c correct to four significant figures.

[6 marks]

Question 9b

b)

According to the model found in part (a) (ii), find

i)

the profit the restaurant is predicted to make on day 100, to the nearest thousand USD,

ii)

the first day for which the loss is less than \$20 000 USD.

[4 marks]**Question 10a**

The primary square root of a complex number z is defined as $\sqrt{z} = x + iy$, where $x, y \in \mathbb{R}$ and $x \geq 0$. If $x = 0$ then the value for y is chosen such that $y \geq 0$. Note that the other square root of z will then be given by $-\sqrt{z} = -x - iy$.

(a)

Show that

$$x = \sqrt{\frac{\operatorname{Re}(z) + \sqrt{(\operatorname{Re}(z))^2 + (\operatorname{Im}(z))^2}}{2}}$$

[5 marks]

Question 10b

(b)

Given that $x > 0$, derive a formula for y in terms of x and $\text{Im}(z)$, and explain why y in this case will always have the same sign (positive, negative, or zero) as $\text{Im}(z)$.

[2 marks]

Question 10c

(c)

Hence show that in general

$$y = \pm \sqrt{\frac{-\text{Re}(z) + \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2}}{2}}$$

with the choice of the positive or negative value being dependent on the properties of z .

[2 marks]

Question 10d

(d)

Explain what must be true of z for each of the following to be true:

(i)

$$x = 0, y \neq 0$$

(ii)

$$x \neq 0, y = 0$$

(iii)

$$x = 0, y = 0$$

[3 marks]

