

4.13 Transition Matrices & Markov Chains

Question Paper

Course	DPIB Maths
Section	4. Statistics & Probability
Topic	4.13 Transition Matrices & Markov Chains
Difficulty	Medium

Time allowed: 80
Score: /62
Percentage: /100

Question 1a

It is known that in the town of Nikudy the weather displays the following patterns:

- If it rains on one day then there is a probability of 0.6 that it will rain on the following day
- If it does not rain on one day then there is only a probability of 0.2 that it will rain on the following day

a)

Represent this information as

(i)

a transition state diagram

(ii)

Indented a transition matrix

[1 mark]

Question 1b

Let T be the transition matrix found in part (a)(ii).

b)

Find T^2 .

[1 mark]

Question 1c

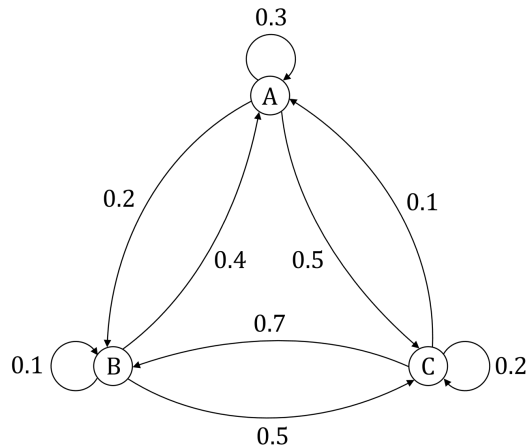
(c)

Hence find the probability that it will rain on Wednesday, given that it did not rain on the preceding Monday. Justify your answer.

[1 mark]

Question 2a

In a robotics lab a robot is programmed to move randomly between three different locations, A , B and C , according to a fixed set of probabilities. At each 'step' of the robot's movement about the lab, the robot will either remain where it is or else move to another location according to the probabilities in the following transition state diagram:



(a)

Write down the transition matrix T for this system of probabilities.

[2 marks]

Question 2b

(b)

Given that the robot begins at location C , find the probabilities that the robot will be at locations A , B or C three steps later.

[3 marks]

Question 2c

(c)

By considering the matrices T^{50} and T^{100} , determine the long-term probabilities of the robot being found at locations A, B or C. State whether or not these probabilities depend on the robot's starting position.

[4 marks]

Question 3a

Two social media influencers, Mememe and EegoTiss, are in a constant struggle to steal each other's followers. No one who follows Mememe will ever follow EegoTiss at the same time, and no one who follows EegoTiss will ever follow Mememe at the same time. Each week, however, 15% of the people who follow Mememe switch to following EegoTiss and 20% of the people who follow EegoTiss switch to following Mememe. It may be assumed that there are no other gains or losses of followers by the two influencers.

a)

Write down a transition matrix T representing the movement of followers between the two influencers in a particular week.

[2 marks]

Question 3b

Initially Mememe and EegoTiss each have 7000 followers.

(b)

(i)

Write down the initial state vector \mathbf{s}_0 for the system.

(ii)

Find the product $\mathbf{T}^5 \mathbf{s}_0$.

(iii)

Hence determine the number of followers that Mememe and EegoTiss will each have after five weeks.

[3 marks]

Question 3c

(c)

Find the number of followers that Mememe and EegoTiss will each have in the long term.

[3 marks]

Question 3d

(d)

Find the total number of followers per week in the long term who will change from following one influencer to following the other.

[2 marks]

Question 4a

In a videogame three mighty wizards – Eugenes (E), Ischyros (I) and Skleros (S) – are attempting to create armies of magical followers. They do this by magically changing members of the other armies into members of their own armies. This happens in the following ways:

- Eugenes' army is made up of unicorns. During each turn of the game he quietly turns 40% of Ischyros' myrmidons and 40% of Skleros' orcs into unicorns.
- Ischyros' army is made up of myrmidons. During each turn of the game he powerfully turns 20% of Eugenes' unicorns and 50% of Skleros' orcs into myrmidons.
- Skleros' army is made up of orcs. During each turn of the game he wickedly turns 20% of Eugenes' unicorns and 50% of Ischyros' myrmidons into orcs.

There is no other way for the numbers of creatures in each of the wizard's armies to increase or decrease.

a)

Write down a transition matrix T representing the changes in the wizards' armies from turn to turn.

[3 marks]

Question 4b

At the start of a particular game Eugenes has 10 unicorns, Ischyros has 80 myrmidons, and Skleros has 350 orcs.

(b)

Find the number of creatures in each wizard's army after one turn.

[3 marks]

Question 4c

(c)

Find the number of creatures that each wizard can expect to have in his army if the game continues for a large number of turns.

[3 marks]

Question 5a

The marketing department of ShedHead brand shampoo ("It makes you look like you woke up in a garden shed!") is attempting to predict the percentage of potential customers who will purchase its product month by month in the future.

One marketing researcher believes that the probability of a potential customer buying ShedHead shampoo one month depends on what shampoo they bought the previous month, as well as what shampoo they bought the month before that.

(a)

Explain why a Markov chain cannot be used to represent month by month sales for this marketing researcher's model.

[1 mark]

Question 5b

Another marketing researcher believes that what shampoo a potential customer purchases one month depends only on what shampoo the customer purchased the previous month. Her research shows that if a customer buys ShedHead shampoo one month then there is a 93% chance they will buy it again the following month, while if a customer does not buy ShedHead shampoo one month then there is only a 5% chance that they will buy it the following month.

(b)

Write down a transition matrix T representing customer behaviour according to this researcher's model.

[2 marks]

Question 5c

Currently 20% of potential customers buy ShedHead shampoo.

(c)

Find the probability that a randomly selected potential customer

(i)

will purchase ShedHead shampoo next month

(ii)

will purchase ShedHead shampoo in the long term.

[5 marks]

Question 6a

A delivery company operates a fleet of lorries serving three major cities, **A**, **B** and **C**. Past experience shows that if a lorry starts a week in city **A** there is a 70% chance that it will still be in city **A** at the start of the following week; otherwise there is a 20% chance that it will be in city **B** and a 10% chance that it will be in city **C**. If a lorry starts a week in city **B** there is an 80% chance that it will still be in city **B** at the start of the following week; otherwise it is equally likely to be in city **A** or city **C**. If a lorry starts a week in city **C** there is a 90% chance that it will still be in city **C** at the start of the following week; otherwise it will be in city **A**, with no chance of it being in city **B**.

(a)

Write down a transition matrix **T** representing the movement of the company's lorries from week to week according to the above information.

[2 marks]

Question 6b

(b)

By solving the system of linear equations represented by

$$T\mathbf{p} = \mathbf{p}$$

determine a steady state vector $\mathbf{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ corresponding to matrix T .

[3 marks]

Question 6c

The company is about to replace its entire fleet of lorries with a fleet of 280 brand new lorries.

(c)

Suggest how the company should initially distribute the new lorries between cities A, B and C. Be sure to justify your answer.

[2 marks]

Question 7a

In the town of Manh, all the residents belong to either one or the other of the town's two fitness clubs – Giang's House of Fitness (G) or Thu's Wonder Gym (T). Each year 30% of the members of G switch to T and 25% of the members of T switch to G. Any other losses or gains of members by the two fitness clubs may be ignored.

a)

Write down a transition matrix T representing the movement of members between the two clubs in a particular year.

[2 marks]

Question 7b

(b)

Find the eigenvalues and corresponding eigenvectors of T .**[4 marks]****Question 7c**

(c)

Hence write down matrices P and D such that $T = PDP^{-1}$.**[2 marks]****Question 7d**

Initially there are 2500 members of G and 800 members of T.

(d)

Using the matrix power formula, show that the numbers of members of G and T after n years will be $(1500 + 1000(0.45^n))$ and $(1800 - 1000(0.45^n))$, respectively.**[6 marks]**

Question 7e

(e)
Hence write down the number of customers that each of the fitness clubs can expect to have in the long term.

[2 marks]