5.11 MacLaurin Series

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.11 MacLaurin Series
Difficulty	Medium

Time allowed: 100

Score: /80

Percentage: /100

Question 1a

Consider the general Maclaurin series formula

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$$

(where $f^{(n)}$ indicates the n^{th} derivative of f).

a)
Use the formula to find the first five terms of the Maclaurin series for e^{2x}.

[4 marks]

Question 1b

b)
Hence approximate the value of e^{2x} when $x = 1$.

[2 marks]

Question 1c

c)
(i)
Compare the approximation found in part (b) to the exact value of e^{2x} when $x = 1$.

(ii)
Explain how the accuracy of the Maclaurin series approximation could be improved.

[3 marks]

Question 1d

d)
Use the general Maclaurin series formula to show that the general term of the Maclaurin series for e^{2x} is

$$\frac{(2x)^n}{n!}$$

[2 marks]

Question 2a

a)
Use substitution into the Maclaurin series for $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

to find the first four terms of the Maclaurin series for $\sin\left(\frac{x}{2}\right)$.

[3 marks]

Question 2b

b)

Hence approximate the value of $\sin\frac{\pi}{2}$ and compare this approximation to the exact value.

[3 marks]

Question 2c

c)
Without performing any additional calculations, explain whether the answer to part (a) would be expected to give an approximation of $\sin\frac{\pi}{4}$ that is more accurate or less accurate than its approximation for $\sin\frac{\pi}{2}$.

[2 marks]

Question 3a

The Maclaurin series for e^x and $\sin x$ are

$$e^x = 1 + x + \frac{x^2}{2!} + \dots \quad \text{and} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

a)
Find the Maclaurin series for $e^x \sin x$ up to and including the term in x^4.

[4 marks]

Question 3b

b)

Use the Maclaurin series for $\sin x$, along with the fact that $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$, to find the first four terms of the Maclaurin series for $\cos x$.

[3 marks]

Question 4a

a)

Use the general Maclaurin series formula to find the first four terms of the Maclaurin series for $\frac{1}{1+x}$.

[4 marks]

Question 4b

b)

Confirm that the answer to part (a) matches the first four terms of the binomial theorem expansion of $\frac{1}{1+x}$.

[3 marks]

Question 4c

The Maclaurin series for $\ln(1+x)$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

c)

Differentiate the Maclaurin series for $\ln(1+x)$ up to its fourth term and compare this to the answer from part (a). Give an explanation for any similarities that are found.

[2 marks]

Question 5a

a)

Use the Maclaurin series for $\sin x$ and $\cos x$ to find a Maclaurin series approximation for $2 \sin x \cos x$ up until the term in x^4.

[3 marks]

Question 5b

The double angle identity for sine tells us that

$$\sin 2x = 2\sin x\cos x$$

b)
Use substitution into the Maclaurin series for $\sin x$ to find a Maclaurin series approximation for $\sin 2x$ up until the term in x^4, and confirm that this matches the answer to part (a).

[3 marks]

Question 6a

a)
Use the Binomial theorem to find a Maclaurin series for the function f defined by

$$f(x) = \sqrt{1 - 2x^2}$$

Give the series up to and including the term in x^6.

[4 marks]

Question 6b

b)
State any limitations on the validity of the series expansion found in part (a).

[2 marks]

Question 6c

c)
Use the answer to part (a) to estimate the value of $\sqrt{0.5}$, and compare the accuracy of that estimated value to the actual value of $\sqrt{0.5}$.

[4 marks]

Question 7a

Consider the differential equation

$$y' = 2y^2 + x$$

together with the initial condition $y(0) = 1$.

a)
(i)
Show that $y'' = 4yy' + 1$.

(ii)
Use an equivalent method to find expressions for y''', $y^{(4)}$ and $y^{(5)}$. Each should be given in terms of y and of lower-order derivatives of y.

[4 marks]

Question 7b

b)
Using the boundary condition above, calculate the values of $y'(0)$, $y''(0)$, $y'''(0)$, $y^{(4)}(0)$ and $y^{(5)}(0)$.

[3 marks]

Question 7c

Let $f(x)$ be the solution to the differential equation above with the given boundary condition, so that $y = f(x)$.

c)
Using the answers to part (b), find the first six terms of the Maclaurin series for $f(x)$.

[4 marks]

Question 7d

d)
Hence approximate the value of to 4 d.p. when $x = 0.1$.

[2 marks]

Question 8a

Consider the differential equation

$$y' = 2xy^2$$

with the initial condition $y(0) = 1$.

a)
(i)
Find y''.

(ii)
Hence show that $y''' = 8yy' + 4x(y')^2 + 4xyy''$ and

$$y^{(4)} = 12(y')^2 + 12yy'' + 12xy'y'' + 4xyy'''$$

[5 marks]

Question 8b

b)

Use the results from part (a) along with the given initial condition to find a Maclaurin series to approximate the solution of the differential equation, giving the approximation up to the term in x^4.

[4 marks]

Question 8c

c)

Use separation of variables to show that the exact solution of the differential equation with the given initial condition is

$$y = \frac{1}{1 - x^2}$$

[4 marks]

Question 8d

d)

Use the binomial theorem to find an approximation for $\frac{1}{1-x^2}$ up to the term in x^4, and verify that it matches the answer to part (b).

[3 marks]