# 1.7 Matrices

## Question Paper

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| Course | DP IB Maths |
| Section | 1. Number & Algebra |
| Topic | 1.7 Matrices |
| Difficulty | Very Hard |

**Time allowed:** 100

**Score:** /78

**Percentage:** /100

## Question 1a

Consider the following matrices:

$$\boldsymbol{A} = \begin{pmatrix} a & 4 & 5 \\ 1 & 2 & 6b \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 1 & 3 \end{pmatrix} \qquad \boldsymbol{C} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

The matrix product $(\boldsymbol{MN})\boldsymbol{P}$ is calculated, where ***M***, ***N***, and ***P*** can each be any one of the matrices ***A***, ***B*** or ***C***.

a)
Find the possible dimensions of the resulting products.

**[4 marks]**

## Question 1b

The matrix sum $\boldsymbol{M} + \boldsymbol{N} + \boldsymbol{P}$ is calculated, where ***M***, ***N***, and ***P*** can each be any one of the matrices ***A***, ***B*** or ***C***.

b)
Find the possible sums that could result from such an addition.

**[2 marks]**

## Question 1c

It is given that $(\boldsymbol{CB})\boldsymbol{A} = t\begin{pmatrix} 2 & 4 & 9.2 \\ 1 & 2 & 4.6 \\ 0 & 0 & 0 \end{pmatrix}$.

c)
Find the values of $a$, $b$, and $t$.

**[5 marks]**

## Question 2a

In this question ***A, B,*** and **C** are arbitrary square matrices.

a)
Prove the following matrix results, stating any necessary assumptions:

(i) If $\boldsymbol{AB} = \boldsymbol{C}$ then $\boldsymbol{B} = \boldsymbol{A}^{-1}\boldsymbol{C}$

(ii) $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

**[7 marks]**

## Question 2b

b)
Using the result from part (a)(ii), simplify $(\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{A}^{-1}$.

**[2 marks]**

## Question 3a

Consider the matrices:

$$\boldsymbol{M} = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$$

$$\boldsymbol{N} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a,\ b,\ c,\ d \in \mathbb{R}$ are constants.

a)
Given that ***M*** and ***N*** are commutative, find an expression for ***N*** in terms of *b* and *d* only.

**[5 marks]**

## Question 3b

Given that the inverse matrix $(\boldsymbol{MN})^{-1}$ exists

(i) determine a relationship that the constants $a, b, c$ and $d$ must satisfy

(ii) find $(\boldsymbol{MN})^{-1}$ in terms of $a, b, c$ and $d$.

**[4 marks]**

## Question 4

Given that

$$\boldsymbol{A} = \begin{pmatrix} 2 & 8 \\ 2 & -4 \end{pmatrix}$$

$$\boldsymbol{A}^{-1}\boldsymbol{BA} = \begin{pmatrix} 2 & 0 \\ 5 & 6 \end{pmatrix}$$

$$\boldsymbol{ABC} = 3\boldsymbol{I}$$

find matrices ***B*** and ***C***.

**[6 marks]**

## Question 5a

For any $2 \times 2$ matrices ***M, A*** or ***B***:

a)
Prove that $\det(k\boldsymbol{M}) = k^2 \det \boldsymbol{M}$, where $k$ is a real constant.

**[3 marks]**

## Question 5b

b)
Prove that $\det(\boldsymbol{AB}) = \det \boldsymbol{A} \times \det \boldsymbol{B}$.

**[5 marks]**

## Question 6

Consider the matrix

$$\boldsymbol{M} = \begin{pmatrix} a & 0 \\ c & -a \end{pmatrix}$$

where $a,\ c \in \mathbb{R}$ . Find expressions for $\boldsymbol{M}^{2k}$ and $\boldsymbol{M}^{2k+1}$ where $k$ is a positive integer.

**[7 marks]**

## Question 7a

Consider the matrix

$$\boldsymbol{A} = \begin{pmatrix} i & 0 \\ 3 & i \end{pmatrix}$$

where $i = \sqrt{-1}$.

a)
(i) Find $\boldsymbol{A}^2$, $\boldsymbol{A}^3$, $\boldsymbol{A}^4$ and $\boldsymbol{A}^5$.

(ii) Hence determine $\boldsymbol{A}^{15}$.

**[4 marks]**

## Question 7b

b)
Find the general term for $\boldsymbol{A}^n$ where $n$ is a positive integer.

**[3 marks]**

## Question 8

Consider the $2 \times 2$ matrix

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Use algebra to find the requirements that must be satisfied by $a,\ b,\ c$ and $d$ in order for $\boldsymbol{M}^2 = \begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix}$ to be true.

**[7 marks]**

## Question 9a

A professional Football team are looking to buy new players.  Their scouts have returned a shortlist containing 23 English, 17 German, 18 Spanish, and 8 Italian players.

The shortlisted players are in the following proportions for each playing position:

- 11% are goalkeepers
- 29% are wingers
- 39% are defenders
- 21% are strikers

A given player only plays in one of the listed positions.

a)
(i)
Write a column matrix, ***N***, representing the numbers of players from each country, and a row matrix, ***P***, containing the proportions of players in each position.

(ii) Hence find the total number of players in each position on the shortlist.

**[5 marks]**

## Question 9b

b)
Explain why it would be incorrect in general to say that the elements of the matrix ***NP*** represent the numbers of players in each position by nationality. For example to say that $(\boldsymbol{NP})_{1,1}$ (the entry in the first row and first column of matrix ***NP***) might represent the number of English goalkeepers.

**[1 mark]**

## Question 10a

A ball is thrown vertically downwards from the top of a cliff, and its position is tracked from when it is first thrown until it hits the ground (at which point it may be assumed that the ball comes instantaneously to rest).

The height of the ball above the ground after $t$ seconds is modelled by the equation $s(t) = at^2 + bt + c$, where $a$, $b$ and $c$ are real constants and the height $s$ is measured in metres. After 1 second, the ball is 175.4 m above the ground; after 5 seconds, its height is 105 m; and after 6 seconds it is 74.4 m.

a)
Set up and solve a matrix equation to find
i)
the height of the cliff
ii)
the time taken for the ball to reach the ground.

**[6 marks]**

## Question 10b

Another experiment studies the motion of another object that only moves in one dimension. A quartic equation of the form $s = at^4 + bt^3 + ct^2 + dt + e$ is used to model the displacement of the object, where $a,\ b,\ c,\ d$ and $e$ are all real constants with $a \neq 0.$

b)

i)

State the number of measurements of the object's displacement at different times that would need to be taken, in order to find the explicit values of the constants $a,\ b,\ c,\ d$ and $e$.

ii)

State the dimensions of the matrices involved in forming a matrix equation, as in part (a), to determine the constants for the quartic model.

**[2 marks]**