4.13 Transition Matrices & Markov Chains

Question Paper

Course	DP IB Maths
Section	4. Statistics & Probability
Topic	4.13 Transition Matrices & Markov Chains
Difficulty	Very Hard

Time allowed: 90

Score: /75

Percentage: /100

Question 1a

Blythe is a chess player whose results in games during tournaments are strongly affected by the result of the immediately preceding game:

- If he wins a game, his increased confidence means that the probabilities for him winning, drawing or losing the next game are 0.38, 0.35 and 0.27 respectively.
- If he draws a game, his feeling of calm means that the probabilities for him winning, drawing or losing the next game are 0.24, 0.57 and 0.19 respectively.
- If he loses a game, his desire for retribution means that the probabilities for him winning, drawing or losing the next game are 0.47, 0 and 0.53 respectively.

a)
Represent this information as

(i) a transition state diagram

(ii) a transition matrix

[4 marks]

Question 1b

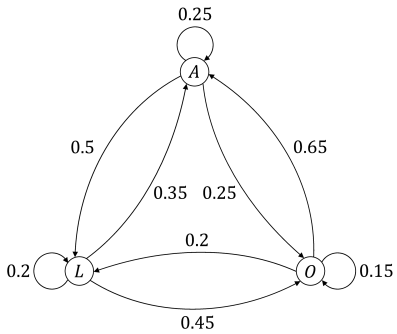
Blythe is playing in a five-day tournament, and he will be playing one game on each of the five days. His game on the last day of the tournament is against his arch-rival Rob Skodur. Blythe does not care how he does in the rest of the tournament, as long as he wins against Rob Skodur on the final day.

b)
Determine the result Blythe should seek in his first game of the tournament, to maximise his chances of winning against Rob Skodur on the final day. Be sure to justify your answer.

[4 marks]

Question 2a

Paul is a pizza fanatic, and he has his dinner every night in one of three pizza restaurants – the Athol House of Pizza (*A*), Lenny and John's Pizzeria (*L*), or Original Pizza II (*O*). Depending on where he has eaten on one night, his choice of restaurant the following night is determined according to the probabilities in the following transition state diagram:



a)
Write down the transition matrix T for this system of probabilities.

[2 marks]

Question 2b

b)
Determine the probabilities that, in the long term,
(i)
Paul will have dinner on Monday night in the same restaurant in which he had dinner on Saturday night
(ii)
where Paul has dinner on Friday night is also where he had dinner on both Wednesday and Thursday nights.

[7 marks]

Question 3a

All members of the Pwll y Felin Mathematics Society are fans of either quaternions or of octonions as being their favourite normed division algebra. At any one time, no quaternions fan will ever also be a fan of octonions, and no octonions fan will ever also be a fan of quaternions. Each week, however, due to ongoing mathematical discoveries, 3.5% of the quaternion fans switch to become octonion fans, and 1.75% of the octonion fans switch to become quaternion fans. It may be assumed that there are no other changes in the numbers of fans of each of the two normed division algebras.

Initially there are 45240 members of the society, and they are evenly split between fans of quaternions and fans of octonions.

a)
Use a matrix method to determine the number of fans that each normed division algebra will have after six weeks.

[4 marks]

Question 3b

b)
Determine the number of society members in the long term who will not change which normed division algebra they are a fan of from any given week to the next.

[4 marks]

Question 4a

The hero Odysseus has landed on the island of Aiaia. The island is ruled by the sorceress Circe who has turned all 600 of Odysseus' followers into beasts. However the goddess Athena has come to help Odysseus, and with her help and his own herbal medicine skills Odysseus hopes to be able to turn his followers back into men and leave the island. Each month on the island:

- Athena turns 15% of the beasts back into men, and turns another 55% of the beasts into half-beasts that will be easier for Odysseus to cure with his herbs
- Odysseus uses his herbs to turn 3% of the beasts and 60% of the half-beasts back to men
- Circe uses her spells to turn 17% of the men and 20% of the half-beasts back into beasts

There is no other way for Odysseus' followers to change between forms, and at any one time each of the followers will be either a 'man' or a 'half-beast' or a 'beast'.

a)
Write down a transition matrix T and an initial state vector s_0 representing the transformations of Odysseus' followers from month to month.

[4 marks]

Question 4b

b)
Show that Odysseus can never get all of his followers turned back into men at the same time.

[3 marks]

Question 4c

Odysseus decides that he will have to leave the island once no more than 130 of his followers are in beast form. Although he will have to leave the beasts behind, he will be able to take all the rest of his followers and cure the half-beasts during the journey home.

c)
Find (i) the whole number of months it will be before Odysseus can leave Aiaia, and (ii) the respective numbers of half-beasts and of men he will have with him on the journey home.

[3 marks]

Question 4d

d)
Determine the long-term fate of Odysseus' followers if Odysseus did not have Athena's help, but all other factors remained the same.

[3 marks]

Question 5a

The marketing department of DeadHead brand shampoo ("It makes you look like you're stuck in a time warp from 1967!") is attempting to predict the percentage of potential customers who will purchase its product month by month in the future.

The lead marketing researcher claims that most of the potential customers for DeadHead shampoo have very short memories, and will not remember what shampoo they have used or how well it has worked for more than a month.

a)
Explain why the lead researcher's claim, if it were true, would support the use of a Markov chain to model DeadHead shampoo's month-on-month market share of potential customers.

[2 marks]

Question 5b

b)
Suggest a counter-claim that, if it were true, would suggest that a Markov chain is *not* a suitable method for modelling DeadHead's month-on-month market share.

[2 marks]

Question 5c

Currently 13% of potential customers buy DeadHead shampoo. Research suggests that there is a 98% chance that a potential customer who buys DeadHead shampoo one month will buy it again the following month. On the other hand, there is only a 5% chance that a potential customer who does not buy DeadHead shampoo one month will switch to buying DeadHead shampoo the following month.

c)
Assuming that a Markov chain may be used to model the situation, find the probability that a randomly selected potential customer

(i) will purchase DeadHead shampoo two months from now

(ii) will purchase DeadHead shampoo in the long term.

[5 marks]

Question 6a

A canoe hire company offers one-day canoe rentals from its three locations in towns on the Millers River – one in Athol (A), one in Orange (O), and one in Millers Falls (M). Past experience shows that two-fifths of canoes hired at A are returned to A; otherwise they are half as likely to be returned to O as they are to M. Only 4% of canoes hired at O are returned to O, with seven of the remainder being returned to M for each one that is returned to A. Of canoes hired at M, 80% are returned to M, with two fifths of the remainder returned to O and the rest returned to A.

a)
By solving an appropriate system of linear equations, determine a steady state vector for the number of canoes at each of the three locations.

[5 marks]

Question 6b

The hire company owns a total of 468 hire canoes. The owner of the company would like to minimise the number of road trips that need to be made to shuttle canoes between the three locations.

b)
Suggest how the owner might distribute the canoes between locations A, O and M in order to minimise the number of road trips. Be sure to justify your answer.

[2 marks]

Question 6c

Athol lies upstream of Orange, which in turn lies upstream of Millers Falls.

c)
Given this additional information, explain why the answer to part (b) might not be the best way for the business to distribute its hire canoes between the three locations.

[2 marks]

Question 7a

In the town of Sodden Chipsbury, all the residents support either one or the other of the town's two lawn bowling teams – the Barnstorming Boulists (B) or the Jackanapes (J). Feelings run high, and supporters of rival teams will sometimes even refuse to chat about the weather with each other. Yet supporters are also fickle, and each year 9% of the supporters of B switch to supporting J and 7% of the supporters of J switch to supporting B. Any other losses or gains of supporters by the two teams may be ignored.

Initially there are 2824 supporters of B and 2232 supporters of J.

a)
Use a matrix method to show that the numbers of supporters of B and J after n years will be $(2212 + 612(0.84^n))$ and $(2844 - 612(0.84^n))$ respectively.

[13 marks]

Question 7b

b)
Hence determine the number of years it will be until the Jackanapes have more supporters than the Barnstorming Boulists.

[3 marks]

Question 7c

c)
Write down the number of supporters each team will have in the long term, being sure to justify your answer

[3 marks]