6.2 Extended Questions (Section B, HL)

Question Paper

Course	DP IB Maths
Section	6. Extended Questions
Topic	6.2 Extended Questions (Section B, HL)
Difficulty	Medium

Time allowed: 110

Score: /90

Percentage: /100

Question 1a

The function f is defined by $f(x) = \dfrac{2x-1}{x^2+3x-4}$, for $x \in \mathbb{R},\ \ x \neq m,\ \ x \neq n.$

a)
Find the values of m and n.

[2 marks]

Question 1b

b)
Find an expression for $f'(x)$.

[3 marks]

Question 1c

The graph of $y = f(x)$ has exactly one point of inflection.

c)
Find the x-coordinate of the point of inflection.

[2 marks]

Question 1d

d)
Sketch the graph of $y = f(x)$ for $-6 \le x \le 6$, showing the coordinates of any axis intercepts and local maxima and local minima, and giving the equations of any asymptotes.

[4 marks]

Question 1e

The function g is defined by $g(x) = \dfrac{x^2 + 3x - 4}{2x - 1}$, for $x \in \mathbb{R},\ x \ne \frac{1}{2}$.

e)
Find the equation of the oblique asymptote of the graph of $y = g(x)$.

[3 marks]

Question 1f

f)
By considering the graph of $y = f(x) - g(x)$, or otherwise, solve $g(x) < f(x)$ for $x \in \mathbb{R}$.

[4 marks]

Question 2a

The function f has a derivative given by $f'(x) = \frac{1}{3x(k-x)}$, $x \in \mathbb{R},\ x \neq 0$, where k is a positive constant.

a)

The expression for $f'(x)$ can be written in the form $\frac{a}{3x} + \frac{b}{k-x}$ where $p,\ q \in \mathbb{R}$. Find a and b in terms of k.

[3 marks]

Question 2b

b)
Hence find an expression for $f(x)$.

[3 marks]

Question 2c

R is the population of rabbits on an island. The rate of change of the population can be modelled by the differential equation $\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{3R(k-R)}{4k}$, where *t* is the time measured in years, $t \geq 0$, and k is the maximum population that the island can support.

The initial population of the rabbits is 20.

c)

By solving the differential equation, show that $R = \frac{20ke^{\frac{3}{4}t}}{k-20+20e^{\frac{3}{4}t}}$

[7 marks]

Question 2d

After two years, the population of rabbits has risen to 70.

d)
Find *k*.

[3 marks]

Question 2e

e)

Find the value of t at which the population of rabbits is growing at its fastest rate.

[2 marks]

Question 3a

A particle is moving in a vertical line and its acceleration, in ms^{-2}, at time t seconds, $t \geq 0$ is given by $a = -\frac{1-v}{2}$, where v is the velocity in meters per second and $v < 1$.

The particle starts at a fixed origin O with initial velocity v_o ms^{-1}.

a)

By solving a suitable differential equation, show that the particle's velocity at time t is given by $v(t) = 1 - e^{-\frac{t}{2}}(1 - v_o)$.

[6 marks]

Question 3b

The particle moves down in the negative direction, until its displacement relative to the origin reaches a minimum. Then the particle changes direction and starts moving up, in a positive direction.

b)
(i)
If the initial velocity of the particle is $-3\ \mathrm{ms}^{-1}$, find the time at which the minimum displacement of the particle from the origin occurs, giving your answer in exact form.

(ii)
If *T* is the time in seconds when the displacement reaches its smallest value, show that $T = 2\ \ln\left(1 - v_o\right)$.

[4 marks]

Question 3c

c)
(i)
Find a general expression for the displacement, in terms of t and v_o.

(ii)
Combine this general expression with the result from part (b)(ii) to find an expression for the minimum displacement of the particle in terms of v_o.

[5 marks]

Question 3d

Let $v(T-k)$ represent the particle's velocity k seconds before the minimum displacement and $v(T+k)$ the particle's velocity k seconds after the minimum displacement.

d)

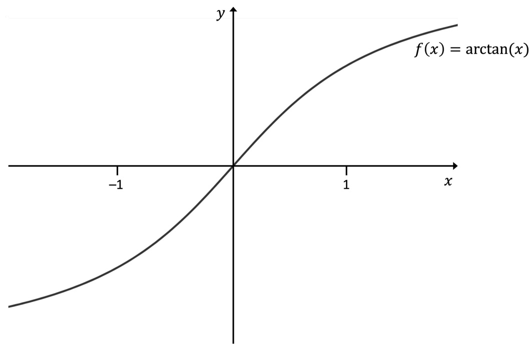
(i) Show that $v(T-k) = 1 - e^{\frac{k}{2}}$.

(ii) Given that $v(T+k) = 1 - e^{-\frac{k}{2}}$, show that $v(T-k) + v(T+k) \geq 0$.

[5 marks]

Question 4a

The diagram below shows the graph of $f(x) = \arctan(x),\ x \in \mathbb{R}$. The graph has rotational symmetry of order 2 about the origin.



a)

A different function, g, is described by $g(x) = -\arctan(x-1),\ x \in \mathbb{R}$.

(i)

Describe the sequence of transformations that transforms $f(x)$ to $g(x)$.

(ii)

Sketch the graph of $g(x)$ on the axes above.

(iii)

Using your answers to parts (i) and (ii) to help you, describe the relationship between $\int_0^1 \arctan(x)\,dx$ and

$\int_0^1 -\arctan(x-1)\,dx$. .

[5 marks]

Question 4b

b)

(i)

Prove that $\arctan p - \arctan q = \arctan\left(\dfrac{p-q}{1+pq}\right)$.

(ii) Show that $\arctan\left(\dfrac{1}{x^2-x+1}\right)$ can be written as $\arctan(x) - \arctan(x-1)$.

[6 marks]

Question 4c

c)

Using the results from parts (a) and (b), evaluate $\int_0^1 \arctan\left(\dfrac{1}{x^2-x+1}\right)\mathrm{d}x$, leaving your answer in exact form.

[7 marks]

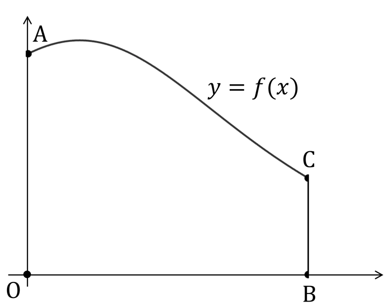
Question 5a

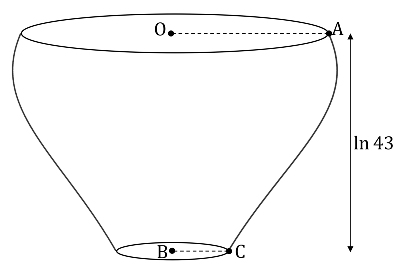
Paola is modelling a small vase from her house for her maths project. To model the edge of the vase in cross-section, she decides to use a function f of the form

$$f(x) = \frac{qe^{\frac{x}{2}}}{2 + e^{x}}$$

where $x \in \mathbb{R},\ x \geq 0$ and $q \in \mathbb{R}^{+}$.

The function and the vase are represented in the diagrams below.





The vertical height of the vase, OB, is measured along the x-axis. The radius of the vase's opening is OA, and its base radius is BC.

To model the vase, she will rotate by 2π radians about the x-axis the region enclosed by the graph of $y = f(x)$, the x-axis, the y-axis, and the line $x = \ln 43$.

a)

Show that the volume of the solid of revolution thus formed is $\frac{14q^{2}\pi}{45}\text{units}^{3}$.

[6 marks]

Question 5b

The volume of the actual vase is 100 cm^3.

b)
Use this information to find the value of q.

[2 marks]

Question 5c

c)
Find the cross-sectional radius of the vase

(i) at its base,

(ii) at its widest point.

[4 marks]

Question 5d

Paola wants to investigate how the cross-sectional radius of the vase changes.

d)
Sketch a graph of the derivative of f, and use it to find the value of x at which the cross-sectional radius of the vase is decreasing most rapidly.

[4 marks]