5.5 Optimisation

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.5 Optimisation
Difficulty	Very Hard

Time allowed: 130

Score: /105

Percentage: /100

Question 1a

A model rocket is released from the ground on a vertical trajectory. The height, h, in metres that it can reach is modelled by

$$h = \frac{2m^2}{15} - \frac{m^3}{480}$$

where m is the mass of fuel in grams.

(a) Find $\frac{dh}{dm}$ and simplify it.

[2 marks]

Question 1b

(b) Using your answer to part (a), find the mass of fuel that is required to reach the maximum height and state what the maximum height is.

[5 marks]

Question 2a

An electric scooter travels between two cities at a speed of x kilometres per hour. Its fuel consumption can be expressed by

$$y = \sqrt{8x} + \frac{25}{x}, \text{ for } x > 1$$

where y is the number of kilowatt hours used.

(a) Find $\frac{dy}{dx}$ and hence the speed that uses the minimum number of kilowatt hours.

[4 marks]

Question 2b

(b) Calculate the minimum amount of electricity that will be consumed for the journey.

[2 marks]

Question 3a

A cone shaped perfume bottle of radius, r, and height, h, is to have all of its external surface area covered in gold leaf. The bottle is required to hold 95 cm^3 of perfume, but the designer wishes to minimise the amount of gold leaf required.

(a) Show that the surface area of the bottle can be expressed as

$$A = \pi r\left(r + \sqrt{\left(\frac{285}{\pi r^2}\right)^2 + r^2}\right)$$

[5 marks]

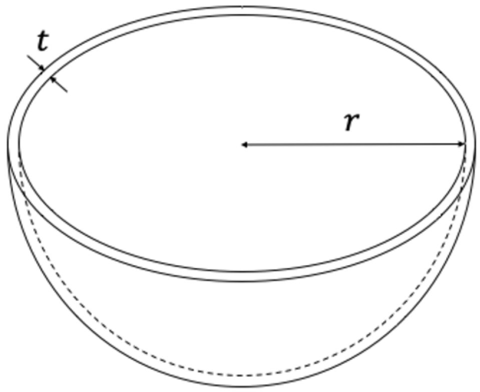
Question 3b

(b) Find the radius of the perfume bottle that will result in the minimum surface area and the minimum amount of gold leaf that is required.

[3 marks]

Question 4a

A decorative bowl is to be created by casting metal in to the form of a hollow hemisphere, with internal radius r cm and thickness t cm, as shown in the diagram below.



A design constraint means that $r + 2t = 25$.

(a) Find the maximum volume of metal that would be required in the construction of the bowl.

[6 marks]

Question 4b

The manufacturer is not happy with the design when the bowl is created using the maximum volume of metal. They claim that, with these dimensions, the volume of metal used in the bowl's construction is 80% more than the capacity of the bowl.

(b) State whether the manufacturer's claim is correct. Justify your answer.

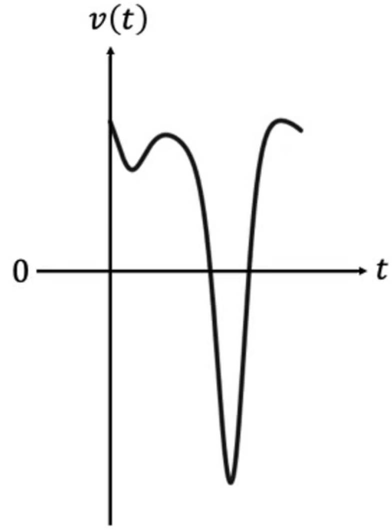
[4 marks]

Question 5a

A particle P moves along a straight line. The velocity v ms^{-1} of P after t seconds is given by

$$v(t) = \frac{1}{2}\sin t - 3t^{\cos t} + 8, \text{ for } 0 \le t \le 10.$$

The following diagram shows the graph of v.



(a) Find the initial velocity of P.

[2 marks]

Question 5b

(b) Find the maximum velocity of P.

[3 marks]

Question 5c

(c) Write down the number of times that the acceleration of P is 0 ms^{-2}.

[3 marks]

Question 5d

(d) Find the acceleration of P when it changes direction the first time.

[4 marks]

Question 6a

A newly-commissioned attack submarine is performing a series of manoeuvres to test its propulsion and steering systems. The vertical position of the submarine relative to sea level (where sea level is represented by $h = 0$) is given by the equation

$$h(t) = 0.0125t^3 - 1.03t^2 + 16.6t - 165, \qquad 0 \le t \le 60$$

where t is the time, in minutes, that has passed since the submarine began its manoeuvres, and $h(t)$ is the vertical position of the submarine in metres.

(a) Find the stationary points for $h(t)$.

[5 marks]

Question 6b

(b) For each of the stationary points found in part (a), determine whether the point is a maximum point or a minimum point. Justify your answer in each case.

[4 marks]

Question 6c

(c) Explain why, in order to find the maximum and minimum depths reached by the submarine in the interval $0 \leq t \leq 60$, it is not sufficient merely to consider the stationary points found in part (a).

[1 mark]

Question 6d

(d) Find the greatest vertical distances that the submarine travels in the interval $0 \leq t \leq 60$ above and below the depth from which it started its manoeuvres.

[2 marks]

Question 7a

Muggins! is a company that produces luxury cribbage boards for discerning collectors of pub game paraphernalia. For sales of between 0 and 100 cribbage boards in a month, the company's profits $P(x)$, in thousands of UK pounds (£1000), can be modelled by the function

$$P(x) = 4.53x^2 - 8.51$$

where x is the number of cribbage boards (in hundreds) sold during the month. For sales of between 100 and 1000 cribbage boards in a month, the corresponding formula is

$$P(x) = 0.02x^3 - \frac{9}{x} + 5$$

Because of manufacturing constraints, the maximum number of cribbage boards that the company can sell in a month is 1000.

(a) (i) Confirm that both formulae give the same profit for sales of 100 cribbage boards in a month.

(ii) State the ranges of x values for which each formula is valid.

[2 marks]

Question 7b

(b) On the same set of axes, sketch the two profit functions. Each function should only be sketched over the interval of x values for which it is valid.

[3 marks]

Question 7c

(c) Show that the combined profit function sketched in part (b) is an increasing function for all valid x values greater than zero.

[4 marks]

Question 7d

(d) Considering only values of x for which $P(x) > 0$, find the value of x for which the instantaneous rate of change of $P(x)$ is a minimum. Give the value of the corresponding instantaneous rate of change, and explain the meaning of that value in context.

[5 marks]

Question 8a

An artist is producing large pieces of sculpture for an art installation. Each piece is in the form of a cylinder with base radius r metres, on top of which is a hemisphere with the same radius as the cylinder's base radius. The hemisphere is fitted exactly to the top of the cylinder, so that the circular bottom of the hemisphere lines up exactly with the circular top of the cylinder.

Every side of each piece of sculpture must be painted, so the artist is eager to find a design for his sculptures such that, for any given volume of a piece of sculpture, the total surface area will be the minimum possible.

(a) Show that for a piece of sculpture with volume $k\pi\ \text{m}^3$, the minimum surface area occurs when

$$r = \sqrt[3]{\frac{3k}{5}}$$

[9 marks]

Question 8b

(b) Find the minimum possible surface area for a piece of sculpture with volume $\frac{40}{3}\pi \text{ m}^3$.

Give your answer as an exact value.

[2 marks]

Question 9a

Two numbers, x and y, are such that $x > y$ and the difference between the two numbers is k, where k is a positive constant.

(a) Find the minimum possible value of the sum $x^2 + 3y^2$, and the values of x and y that correspond to that minimum value. Your answers should be given in terms of k.

[6 marks]

Question 9b

(b) Justify that your answer in part (a) is a minimum value.

[4 marks]

Question 10a

Donty is a would-be social media celebrity who is obsessed with the number of 'likes' his posts receive. He hires a statistician to study his social media accounts, and after analysing several years of data she determines that the rate of change of his number of 'likes' can be modelled by the equation

$$\frac{dL}{dx} = -0.164x^3 + 2.73x^2 - 12.7x + 15.3, \quad 0 \leq x \leq 12$$

where L represents the number of likes received on a given day (in thousands of likes), and x represents the amount of new video content Donty uploaded on the preceding day (in hours). Because of technical limitations, Donty is unable to upload more than 12 hours of new video content on any given day.

It is known as well that 36075 'likes' are received on a day after 5 hours of video content was uploaded the day before.

(a) Find the maximum and minimum number of 'likes' that Donty can expect to receive in a day, and the corresponding number of hours of new video content that Donty should upload on the preceding day to attain that maximum or minimum. Be sure to justify that the values you find are indeed the maximum and minimum possible.

[11 marks]

Question 10b

(b) (i) For the maximum value determined in part (a), calculate the number of likes that are received for each minute of new video content uploaded the preceding day.

(ii) State, with a reason, whether the value calculated in part (b) (i) represents the maximum number of 'likes per minute of new content' that Donty is able to achieve.

[4 marks]