5.8 Advanced Differentiation

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.8 Advanced Differentiation
Difficulty	Medium

Time allowed: 100

Score: /80

Percentage: /100

Question 1

Let $f(x) = 3x^2$.

By differentiating from first principles, show that $f'(x) = 6x$.

[4 marks]

Question 2a

Let $f(x) = \sin x$.

a)
Solve the equation $f'(x) = f'''(x)$ in the interval $0 \leq x \leq 2\pi$.

[1 mark]

Question 2b

b)
Show that $f^{(4)}(x) = f(x)$.

[2 marks]

Question 3a

Find the derivative of each of the following functions:

a)

$$f(x) = \cot\left(x + \frac{\pi}{3}\right)$$

[2 marks]

Question 3b

b)
$g(x) = 5^x - 3\log_3 x$

[2 marks]

Question 3c

c)
$h(x) = \arcsin 4x$

[3 marks]

Question 4a

a)
For the curve defined by $y = \tan 3x$, show that

$$\frac{d^2y}{dx^2} = 18(\tan 3x + \tan^3 3x)$$

[4 marks]

Question 4b

b)
For the curve defined by $y = \arctan x$, show that

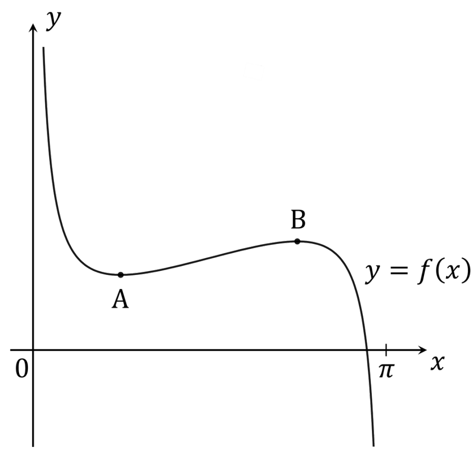
$$y'' = -\frac{2x}{x^4 + 2x^2 + 1}$$

[2 marks]

Question 5a

Consider the function f defined by $f(x) = 2x + \cot x,\ 0 < x < \pi$.

The following diagram shows the graph of the curve $y = f(x)$:



The points marked A and B are the turning points of the graph.

a)
(i)
Find $f'(x)$.

(ii)
Hence find the coordinates of points A and B.

[6 marks]

Question 5b

b)

Find the equation of the normal to the graph at the point where the x-coordinate is equal to $\frac{\pi}{2}$.

[4 marks]

Question 6a

For each of the following, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by differentiating implicitly with respect to x.

a)
$x^2 + y^2 = 16$

[2 marks]

Question 6b

b)
$4x^2 - 3x = y^2 + 2y$

[2 marks]

Question 6c

c)

$$\frac{(x+y)^2}{3x}=1$$

[3 marks]

Question 6d

d)

$$\sqrt{x^2+y^3}=4$$

[4 marks]

Question 7a

A curve is described by the equation

$$\frac{2}{x} - \frac{1}{y} = 1$$

a)
Use implicit differentiation with respect to x to show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y^2}{x^2}$$

[2 marks]

Question 7b

b)
Use your result from part (a) to find the equation of the

(i)
tangent

(ii)
normal

to the curve at the point $(1,\ 1)$.

[4 marks]

Question 7c

(c)

(i)

Rearrange the equation of the curve into the form $y = f(x)$.

(ii)

Hence find an expression for $\frac{dy}{dx}$ entirely in terms of x.

[5 marks]

Question 7d

d)

Verify that your answer to part (c)(ii) and the result from part (b)(i) both give the same value for the gradient of the tangent to the curve at the point $(1, 1)$.

[2 marks]

Question 8a

An international mission has landed a rover on the planet Mars. After landing, the rover deploys a small drone on the surface of the planet, then rolls away to a distance of 6 metres in order to observe the drone as it lifts off into the air. Once the rover has finished moving away, the drone ascends vertically into the air at a constant speed of 2 metres per second.

Let D be the distance, in metres, between the rover and the drone at time t seconds.

Let h be the height, in metres, of the drone above the ground at time t seconds. The entire area where the rover and drone are situated may be assumed to be perfectly horizontal.

a)
Show that

$$D = \sqrt{h^2 + 36}$$

[2 marks]

Question 8b

b)
(i)
Explain why $\frac{\mathrm{d}h}{\mathrm{d}t} = 2$.

(ii)
Hence use implicit differentiation to show that

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{2h}{\sqrt{h^2 + 36}}$$

[5 marks]

Question 8c

c)
Find

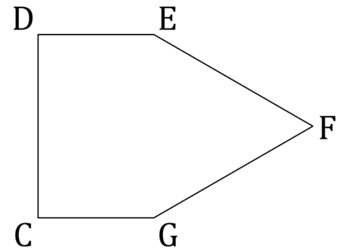
(i)
the rate at which the distance between the rover and the drone is increasing at the moment when the drone is 8 metres above the ground.

(ii)
the height of the drone above the ground at the moment when the distance between the rover and the drone is increasing at a rate of $1\ \text{ms}^{-1}$.

[4 marks]

Question 9a

In the diagram below, CDEFG is the outline of a type of informational signboard that a county council plans to use in one of its parks. The shape is formed by a rectangle CDEG, to one side of which an equilateral triangle EFG has been appended.



The signboards will be produced in various different sizes. However because of the cost of the edging that must go around the perimeter of the signboards, the council is eager to design the signboards so that the area of a signboard is the maximum possible for a given perimeter.

Let $|CD| = x$ cm and let $|DE| = y$ cm.

a)

(i)

Write down an expression in terms of x and y for the perimeter of the signboard, P.

(ii)

Hence use implicit differentiation to find $\frac{dP}{dx}$

[3 marks]

Question 9b

b)

Explain why, for a given perimeter, it must be true that $\frac{dP}{dx} = 0$, and use this fact to show that $\frac{dy}{dx} = -\frac{3}{2}$.

[3 marks]

Question 9c

c)

Show that the area A, of the signboard is given by $A = xy + \frac{\sqrt{3}}{4}x^2$.

[4 marks]

Question 9d

d)
Hence use implicit differentiation to find the ratio of y to x that gives the maximum area.

[5 marks]