5.9 Advanced Integration

Question Paper

Course	DP IB Maths
Section	5. Calculus
Topic	5.9 Advanced Integration
Difficulty	Hard

Time allowed: 130

Score: /101

Percentage: /100

Question 1a

Find the following indefinite integrals:

(a)

$$\int 3\ \mathrm{cosec}^2\frac{\pi x}{2}\,\mathrm{d}x$$

[2 marks]

Question 1b

(b)

$$\int \frac{\tan 2x}{\cos 2x}\,\mathrm{d}x$$

[3 marks]

Question 1c

(c)

$$\int \frac{\tan^2\left(x-\frac{\pi}{3}\right)}{1-\cos^2\left(x-\frac{\pi}{3}\right)}\,\mathrm{d}x$$

[3 marks]

Question 2a

Find the following indefinite integrals:

(a)

$$\int (\ln 4) 2^x \, \mathrm{d}x$$

[3 marks]

Question 2b

(b)

$$\int \frac{3}{9x^2 + 4} \, \mathrm{d}x$$

[3 marks]

Question 2c

(c)

$$\int \frac{2\pi}{\sqrt{(x+5)(5-x)}} \, \mathrm{d}x$$

[3 marks]

Question 2d

(d)
Using a sketch, briefly describe the family of graphs corresponding to all the possible specific solutions to the integral in part (a).

[3 marks]

Question 3a

(a)
Show that

$$\frac{8}{x^2+2x-15} = \frac{A}{x-3} + \frac{B}{x+5}$$

where A and B are constants to be found.

[2 marks]

Question 3b

(b)
Hence show that

$$\int_5^8 \frac{8}{x^2+2x-15}\,\mathrm{d}x = 2\ln 5 - \ln 13$$

[5 marks]

Question 4

Use the substitution $u = \sin x + 1$ to find the exact value of the following definite integral:

$$\int_0^{\frac{\pi}{6}} \sin 2x(1 + \sin x)^2 \mathrm{d}x$$

[7 marks]

Question 5a

(a)

Show that $x^2 + 6x + 18$ may be written in the form $p + (x - q)^2$, where p and q are constants to be determined.

[2 marks]

Question 5b

(b)

Hence find the exact value of the definite integral

$$\int_{-6}^{0} \frac{1}{x^2 + 6x + 18} \mathrm{d}x$$

[7 marks]

Question 6a

(a)

Show that

$$\frac{u}{2+u} = A + \frac{B}{2+u}$$

where A and B are constants to be determined.

[2 marks]

Question 6b

(b)

Use the substitution $u = \sqrt{x+7}$ along with the result from part (a) to find the indefinite integral

$$\int \frac{3}{2+\sqrt{x+7}}\,\mathrm{d}x$$

giving your answer as a function of x.

[5 marks]

Question 7a

(a)

Use integration by parts to show that

$$\int \ln x \, \mathrm{d}x = x \ln x - x + c$$

where c is a constant of integration.

[4 marks]

Question 7b

(b)

Hence find the exact value of the definite integral

$$\int_1^{e^3} \ln x \, \mathrm{d}x$$

being sure to simplify your answer as far as possible.

[3 marks]

Question 8a

(a)

Show that

$$\int ax \cos bx \, \mathrm{d}x = \frac{a}{b} x \sin bx + \frac{a}{b^2} \cos bx + c$$

where $a, b \in \mathbb{R}$ are non-zero constants, and c is a constant of integration.

[4 marks]

Question 8b

(b)

A continuous random variable X has the probability density function f given by

$$f(x) = \begin{cases} \frac{x}{\pi - 2} \cos\left(\frac{x}{\sqrt{2}}\right), & 0 \leq x \leq \frac{\pi}{\sqrt{2}} \\ 0, & \text{otherwise} \end{cases}$$

Find $P\left(0 \leq X \leq \frac{\pi\sqrt{2}}{6}\right)$.

[3 marks]

Question 9a

Consider the integral I defined by

$$I = \int e^x \cos x \, \mathrm{d}x$$

(a)
Use integration by parts to show that

$$I = e^x(\sin x + \cos x) - I$$

[6 marks]

Question 9b

(b)
Hence find the indefinite integral

$$\int e^x \cos x \, \mathrm{d}x$$

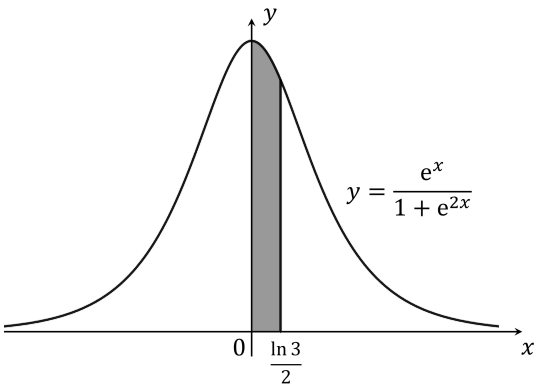
[2 marks]

Question 10

Let f be the function defined by

$$f(x) = \frac{e^x}{1 + e^x}$$

The diagram below shows a part of the graph of the curve $y = f(x)$. The shaded region is the region bounded by the curve, the positive x- and y-axes and the line $x = \frac{\ln 3}{2}$.

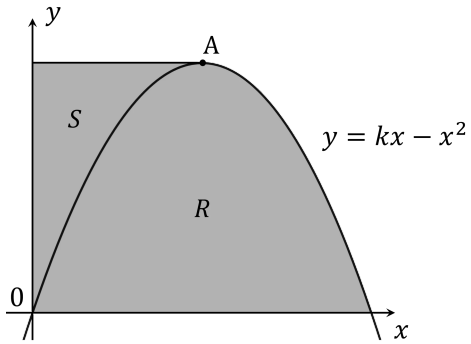


Using the substitution $u = e^x$, or otherwise, find the exact area of the shaded region.

[7 marks]

Question 11a

The following diagram shows a part of the graph of the curve $y = kx - x^2$, where $k > 0$ is a constant. The point marked A is the vertex of the curve. Region R is the region enclosed by the curve and the x-axis. Region S is the region enclosed by the curve, the positive y-axis, and the line through point with gradient zero.



(a)
Show that the part of the curve bordering the region S can also be represented by the curve with equation $x = \frac{1}{2}\left(k - \sqrt{k^2 - 4y}\right)$.

[3 marks]

Question 11b

When region R is rotated 2π radians about the x-axis, the resultant solid of revolution has a volume equal to $\frac{1296\pi}{5}\text{units}^3$.

(b)
Find the value of k.

[5 marks]

Question 11c

(c)
Use the result from part (a) to find the exact area of region S.

[5 marks]

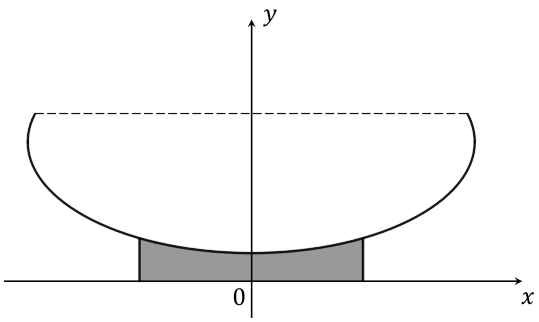
Question 11d

(d)
Use your answer to part (c) to write down the area of region R.

[2 marks]

Question 12

The diagram below shows the cross-section of a miniature goldfish bowl produced by Some Things Fishy, a specialist company supplying products for miniature goldfish enthusiasts.



The glass part of the bowl sits on a solid base, indicated by the shaded region on the diagram. The cross-section of the glass part of the bowl is symmetrical about the y-axis, and may be described by the curve with equation

$$\frac{x^2}{64} + \frac{(y-5)^2}{16} = 1$$

The dashed horizontal line represents the diameter of the open top of the fishbowl. All coordinates are expressed in centimetres, and for purposes of answering this question the thickness of the glass sides of the bowl may be regarded as negligible.

Given that the diameter of the open top of the fishbowl is 15 cm, and that this is less than the diameter of the fishbowl at its widest point, find the capacity of the glass part of the fishbowl.

[7 marks]