Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2018

Mathematics

Higher level

Paper 2

19 pages

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document **"Mathematics HL: Guidance for e-marking November 2018"**. It is essential that you read this document before you start marking. In particular, please note the following.

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(= 10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. (a) $u_4 = u_1 r^3 \Rightarrow -2.916 = 4r^3$ ***(A1)***

solving, $r = -0.9$ ***(M1)A1***

[3 marks]

(b) $S_\infty = \dfrac{4}{1-(-0.9)}$ ***(M1)***

$= \dfrac{40}{19} (= 2.11)$ ***A1***

[2 marks]

Total [5 marks]

2. $f'(x) = \int \left(15\sqrt{x} + \dfrac{1}{(x+1)^2} \right) \mathrm{d}x = 10x^{\frac{3}{2}} - \dfrac{1}{x+1} \ (+c)$ ***(M1)A1A1***

Note: ***A1*** for first term, ***A1*** for second term. Withhold one ***A1*** if extra terms are seen.

$f(x) = \int \left(10x^{\frac{3}{2}} - \dfrac{1}{x+1} + c \right) \mathrm{d}x = 4x^{\frac{5}{2}} - \ln(x+1) + cx + d$ ***A1***

Note: Allow FT from incorrect $f'(x)$ if it is of the form $f'(x) = Ax^{\frac{3}{2}} + \dfrac{B}{x+1} + c$.

Accept $\ln|x+1|$.

attempt to use at least one boundary condition in their $f(x)$ ***(M1)***

$x = 0, y = -4$

$\Rightarrow d = -4$ ***A1***

$x = 1, y = 0$

$\Rightarrow 0 = 4 - \ln 2 + c - 4$

$\Rightarrow c = \ln 2 (= 0.693)$ ***A1***

$f(x) = 4x^{\frac{5}{2}} - \ln(x+1) + x\ln 2 - 4$

[7 marks]

3. (a) use of inverse normal (implied by $\pm 0.1509\ldots$ or $\pm 1.554\ldots$) ***(M1)***

$\mathrm{P}(X < 16) = 0.56$

$\Rightarrow \frac{16 - \mu}{\sigma} = 0.1509\ldots$ ***(A1)***

$\mathrm{P}(X < 17) = 0.94$

$\Rightarrow \frac{17 - \mu}{\sigma} = 1.554\ldots$ ***(A1)***

attempt to solve a pair of simultaneous equations ***(M1)***

$\mu = 15.9,\ \sigma = 0.712$ ***A1A1***

[6 marks]

(b) correctly shaded diagram or intent to find $\mathrm{P}(X \geq 15)$ ***(M1)***

$= 0.895$ ***A1***

Note: Accept answers rounding to 0.89 or 0.90. Award ***M1A0*** for the answer 0.9.

[2 marks]

Total [8 marks]

4. METHOD 1

$\left(x + \frac{3}{x^2}\right)^5 = \ldots + \binom{5}{2} x^2 \left(\frac{3}{x^2}\right)^3 + \ldots$ ***(M1)(A1)(A1)***

Note: Award ***M1*** for a product of a binomial coefficient, a power of x, and a power of $\frac{3}{x^2}$, ***A1*** for correct binomial coefficient, ***A1*** for correct powers.

$= \ldots + 10 \times \frac{27}{x^4} + \ldots \left(= \ldots + \frac{270}{x^4} + \ldots\right)$ ***(A1)***

constant term is $x^4\left(\frac{270}{x^4}\right)$

$= 270$ ***A1***

continued…

Question 4 continued

METHOD 2

EITHER

the general term is $x^4\binom{5}{r}x^r\left(\frac{3}{x^2}\right)^{5-r}$ ***(M1)(A1)***

Note: Award ***M1*** for a product of a binomial coefficient, power(s) of x, and a power of $\frac{3}{x^2}$.

$=\binom{5}{r}\times 3^{5-r}\times\frac{x^{r+4}}{x^{10-2r}}\left(=\binom{5}{r}\times 3^{5-r}x^{3r-6}\right)$

constant term occurs when $r=2$ ***(A1)***

OR

the general term is $x^4\binom{5}{5-r}x^{5-r}\left(\frac{3}{x^2}\right)^{r}$ ***(M1)(A1)***

Note: Award ***M1*** for a product of a binomial coefficient, power(s) of x, and a power of $\frac{3}{x^2}$.

$=\binom{5}{5-r}\times 3^{r}\times\frac{x^{9-r}}{x^{2r}}\left(=\binom{5}{5-r}\times 3^{r}x^{9-3r}\right)$

constant term occurs when $r=3$ ***(A1)***

continued…

Question 4 continued

THEN

$\binom{5}{2}(3)^3$ ***(A1)***

$= 270$ ***A1***

[5 marks]

5. METHOD 1

$\frac{f(x+h)-f(x)}{h}$

$= \frac{\left(3(x+h)^3-(x+h)\right)-\left(3x^3-x\right)}{h}$ ***M1***

$= \frac{3\left(x^3+3x^2h+3xh^2+h^3\right)-x-h-3x^3+x}{h}$ ***(A1)***

$= \frac{9x^2h+9xh^2+3h^3-h}{h}$ ***A1***

cancelling h ***M1***

$= 9x^2+9xh+3h^2-1$

then $\lim_{h\to 0}\left(9x^2+9xh+3h^2-1\right)$

$= 9x^2-1$ ***A1***

Note: Final ***A1*** dependent on all previous marks.

METHOD 2

$\frac{f(x+h)-f(x)}{h}$

$= \frac{\left(3(x+h)^3-(x+h)\right)-\left(3x^3-x\right)}{h}$ ***M1***

$= \frac{3\left((x+h)^3-x^3\right)+\left(x-(x+h)\right)}{h}$ ***(A1)***

$= \frac{3h\left((x+h)^2+x(x+h)+x^2\right)-h}{h}$ ***A1***

cancelling h ***M1***

$= 3\left((x+h)^2+x(x+h)+x^2\right)-1$

then $\lim_{h\to 0}\left(3\left((x+h)^2+x(x+h)+x^2\right)-1\right)$

$= 9x^2-1$ ***A1***

Note: Final ***A1*** dependent on all previous marks.

[5 marks]

6. (a) attempt to substitute $x = 5$ and set equal to zero, or use of long / synthetic division ***(M1)***

$2 \times 5^4 - 15 \times 5^3 + a \times 5^2 + 5b + c = 0$ ***A1***

$(\Rightarrow 25a + 5b + c = 625)$

[2 marks]

(b) 0 ***A1***

[1 mark]

(c) **EITHER**

attempt to solve $P'(5) = 0$ ***(M1)***

$\Rightarrow 8 \times 5^3 - 45 \times 5^2 + 4 \times 5 + b = 0$

OR

$\left(x^2 - 10x + 25\right)\left(2x^2 + \alpha x + \beta\right) = 2x^4 - 15x^3 + 2x^2 + bx + c$ ***(M1)***

comparing coefficients gives $\alpha = 5, \beta = 2$

THEN

$b = 105$ ***A1***

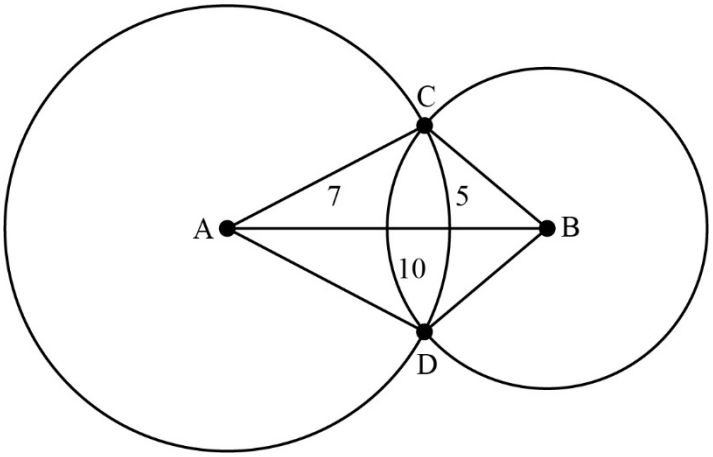
$\therefore c = 625 - 25 \times 2 - 525$

$c = 50$ ***A1***

[3 marks]

Total [6 marks]

7.



use of cosine rule ***(M1)***

$$\hat{\text{CAB}} = \arccos\left(\frac{49 + 100 - 25}{2 \times 7 \times 10}\right) = 0.48276\ldots (= 27.660...^{\circ}) \quad \textit{(A1)}$$

$$\hat{\text{CBA}} = \arccos\left(\frac{25 + 100 - 49}{2 \times 5 \times 10}\right) = 0.70748\ldots (= 40.535...^{\circ}) \quad \textit{(A1)}$$

attempt to subtract triangle area from sector area ***(M1)***

$$\text{area} = \frac{1}{2} \times 49\left(2\hat{\text{CAB}} - \sin 2\hat{\text{CAB}}\right) + \frac{1}{2} \times 25\left(2\hat{\text{CBA}} - \sin 2\hat{\text{CBA}}\right)$$

$$= 3.5079... + 5.3385... \quad \textit{(A1)}$$

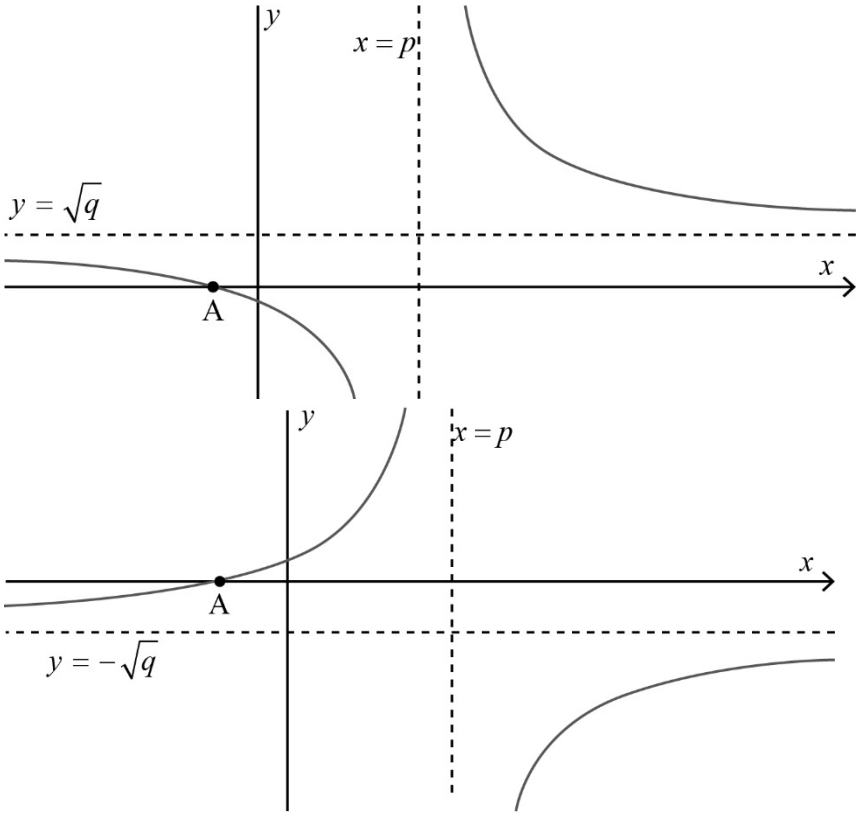
Note: Award this ***A1*** for either of these two values.

$$= 8.85\left(\text{km}^2\right) \quad \textit{A1}$$

Note: Accept all answers that round to 8.8 or 8.9.

[6 marks]

8. (a)



either graph passing through (or touching) A ***A1***
correct shape and vertical asymptote with correct equation for either graph ***A1***
correct horizontal asymptote with correct equation for either graph ***A1***
two completely correct sketches ***A1***
[4 marks]

(b) $a\left(-\frac{1}{2}\right)+1=0 \Rightarrow a=2$ ***A1***

from horizontal asymptote, $\left(\frac{a}{b}\right)^2=\frac{4}{9}$ ***(M1)***

$\frac{a}{b}=\pm\frac{2}{3} \Rightarrow b=\pm 3$ ***A1***

from vertical asymptote, $b\left(\frac{4}{3}\right)+c=0$

$b=3,\ c=-4$ or $b=-3,\ c=4$ ***A1***
[4 marks]

Total [8 marks]

Section B

9. (a) **METHOD 1**

$$f'(x) = \frac{\frac{2(x-3)}{x} - (2\ln x + 1)}{(x-3)^2} \left(= \frac{2(x-3) - x(2\ln x + 1)}{x(x-3)^2} \right) \quad \textit{(M1)A1A1A1}$$

Note: Award ***M1*** for attempt at quotient rule, ***A1A1*** for numerator and ***A1*** for denominator.

METHOD 2

$$f(x) = (2\ln x + 1)(x-3)^{-1} \quad \textit{(A1)}$$

$$f'(x) = \left(\frac{2}{x}\right)(x-3)^{-1} - (2\ln x + 1)(x-3)^{-2} \left(= \frac{2(x-3) - x(2\ln x + 1)}{x(x-3)^2} \right) \quad \textit{(M1)A1A1}$$

Note: Award ***M1*** for attempt at product rule, ***A1*** for first term, ***A1*** for second term.

[4 marks]

(b) finding turning point of $y = f'(x)$ or finding root of $y = f''(x)$ ***(M1)***

$x = 0.899$ ***A1***

$y = f(0.899048\ldots) = -0.375$ ***(M1)A1***

$(0.899, -0.375)$

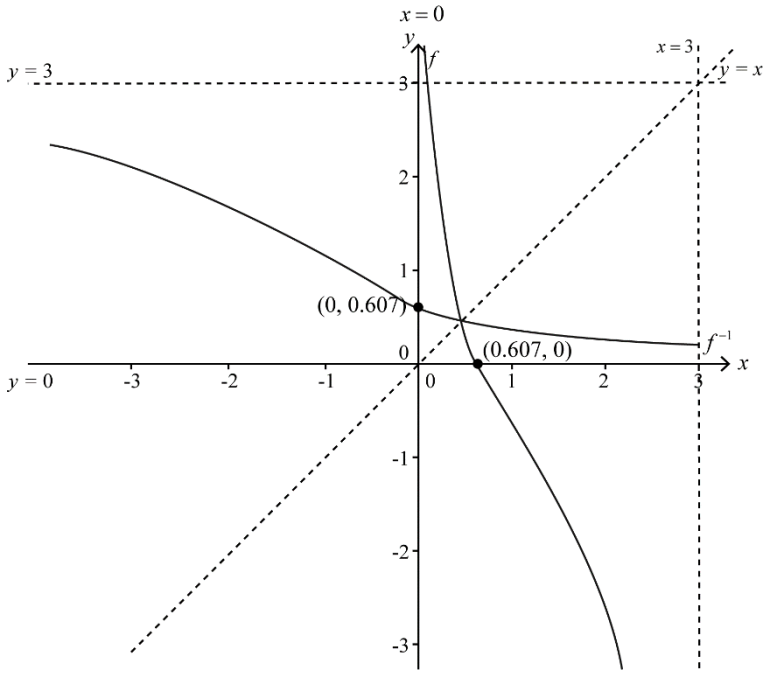
Note: Do not accept $x = 0.9$. Accept y-coordinates rounding to -0.37 or -0.375 but not -0.38.

[4 marks]

continued…

Question 9 continued

(c)



(i) smooth curve over the correct domain which does not cross the y-axis and is concave down for $x > 1$ ***A1***

x-intercept at 0.607 ***A1***

equations of asymptotes given as $x = 0$ and $x = 3$ (the latter must be drawn) ***A1A1***

[4 marks]

(ii) attempt to reflect graph of f in $y = x$ ***(M1)***

smooth curve over the correct domain which does not cross the x-axis and is concave down for $y > 1$ ***A1***

y-intercept at 0.607 ***A1***

equations of asymptotes given as $y = 0$ and $y = 3$ (the latter must be drawn) ***A1***

Note: For ***FT*** from (i) to (ii) award max ***M1A0A1A0***.

[4 marks]

(d) solve $f(x) = f^{-1}(x)$ or $f(x) = x$ to get $x = 0.372$ ***(M1)A1***

$0 < x < 0.372$ ***A1***

Note: Do not award ***FT*** marks.

[3 marks]

Total [19 marks]

10. (a) (i) $\mathrm{P}(X < 60)$

$= \mathrm{P}(X \leq 59)$ ***(M1)***

$= 0.102$ ***A1***

(ii) standard deviation $= \sqrt{70}\ (= 8.37)$ ***(M1)A1***

[4 marks]

(b) (i) use of midpoints (accept consistent use of $45, 55$ etc.) ***(M1)***

$$\frac{44.5 \times 2 + 54.5 \times 15 + 64.5 \times 40 + 74.5 \times 53 + 94.5 + 104.5 \times 3 + 114.5 \times 6}{2 + 15 + 40 + 53 + 0 + 1 + 3 + 6}$$

(M1)

$= \frac{8530}{120} (= 71.1)$ ***A1***

Note: If 45, 55, etc. are used consistently instead of midpoints (implied by the answer 71.58…) award ***M1M1A0.***

(ii) 13.9 ***(M1)A1***

[5 marks]

(c) valid reason given to include the examples below ***R1***
variance is 192 which is not close to the mean (accept not equal to)
standard deviation too high (using parts (a)(ii) and (b)(ii))
relative frequency of $X \leq 59$ is 0.142 which is too high (using part (a)(i))
Poisson would give a frequency of roughly 14 for $80 \leq X \leq 89$

Note: Reasons which do not use values found in previous parts must be backed up with numerical evidence.

[1 mark]

(d) $\mathrm{P}(Y > 10) = 0.99$

$1 - \mathrm{P}(Y \leq 10) = 0.99 \Rightarrow \mathrm{P}(Y \leq 10) = 0.01$ ***(M1)***

attempt to solve a correct equation ***(M1)***

$\lambda = 20.1$ ***A1***

[3 marks]

continued…

Question 10 continued

(e) in 1 day, no of emails is $X \sim \text{Po}(\lambda)$

in 2 days, no of emails is $Y \sim \text{Po}(2\lambda)$ ***(A1)***

$\text{P}(10 \text{ on first day} \mid 20 \text{ in 2 days})$ ***(M1)***

$= \dfrac{\text{P}(X = 10) \times \text{P}(X = 10)}{\text{P}(Y = 20)}$ ***(M1)***

$= \dfrac{\left(\dfrac{\lambda^{10} e^{-\lambda}}{10!}\right)^2}{\dfrac{(2\lambda)^{20} e^{-2\lambda}}{20!}}$ ***A1***

$= \dfrac{\lambda^{20} e^{-2\lambda}}{2^{20} \lambda^{20} e^{-2\lambda}} \times \dfrac{20!}{(10!)^2}$ ***A1***

$= \dfrac{20!}{2^{20}(10!)^2}$

which is independent of λ ***AG***

[5 marks]

Total [18 marks]

11. (a) **METHOD 1**

use of tan ***(M1)***

$\tan\theta_p = \dfrac{1}{p}$ ***(A1)***

$\theta_p = \arctan\left(\dfrac{1}{p}\right)$ ***A1***

METHOD 2

$\text{AP} = \sqrt{p^2 + 1}$ ***(A1)***

use of sin, cos, sine rule or cosine rule using the correct length of AP ***(M1)***

$\theta_p = \arcsin\left(\dfrac{1}{\sqrt{p^2+1}}\right)$ or $\theta_p = \arccos\left(\dfrac{p}{\sqrt{p^2+1}}\right)$ ***A1***

[3 marks]

continued…

Question 11 continued

(b) $\text{QR} = 1 \Rightarrow r = q + 1$ ***(A1)***

Note: This may be seen anywhere.

$\tan\theta_p = \tan\left(\theta_q + \theta_r\right)$

attempt to use compound angle formula for tan ***M1***

$\tan\theta_p = \dfrac{\tan\theta_q + \tan\theta_r}{1 - \tan\theta_q \tan\theta_r}$ ***(A1)***

$\dfrac{1}{p} = \dfrac{\dfrac{1}{q} + \dfrac{1}{r}}{1 - \left(\dfrac{1}{q}\right)\left(\dfrac{1}{r}\right)}$ ***(M1)***

$\dfrac{1}{p} = \dfrac{\dfrac{1}{q} + \dfrac{1}{q+1}}{1 - \left(\dfrac{1}{q}\right)\left(\dfrac{1}{q+1}\right)}$ or $p = \dfrac{1 - \left(\dfrac{1}{q}\right)\left(\dfrac{1}{q+1}\right)}{\dfrac{1}{q} + \dfrac{1}{q+1}}$ ***A1***

$\dfrac{1}{p} = \dfrac{q + q + 1}{q(q+1) - 1}$ ***M1***

Note: Award ***M1*** for multiplying top and bottom by $q(q+1)$.

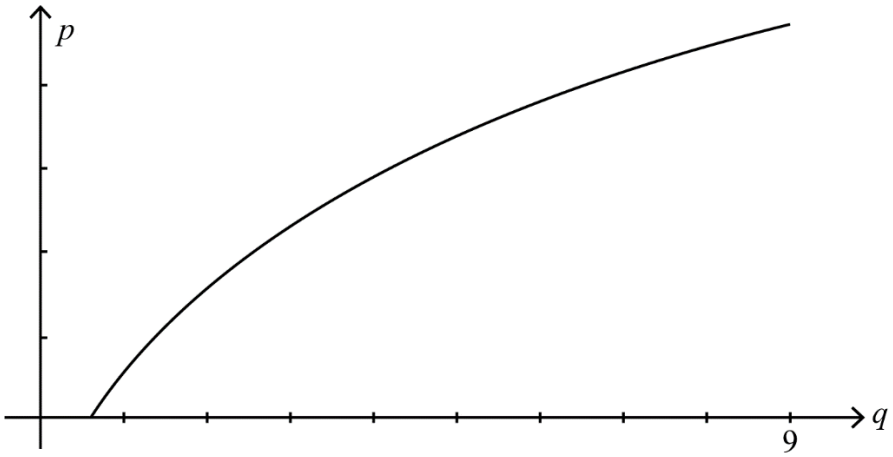
$p = \dfrac{q^2 + q - 1}{2q + 1}$ ***AG***

[6 marks]

continued…

Question 11 continued

(c)



increasing function with positive q-intercept ***A1***

Note: Accept curves which extend beyond the domain shown above.

$(0.618 <) \; q < 9$ ***(A1)***

$\Rightarrow$ range is $(0 <) \; p < 4.68$ ***(A1)***

$0 < p < 4.68$ ***A1***

[4 marks]

Total [13 marks]