N13/5/MATHL/HP3/ENG/TZ0/DM/M

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

MARKSCHEME

November 2013

MATHEMATICS
DISCRETE MATHEMATICS

Higher Level

Paper 3

15 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to Scoris instructions and the document "**Mathematics HL: Guidance for e-marking November 2013"**. It is essential that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by Scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets, for example, (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a mis-read. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) use Kruskal's algorithm: begin by choosing the shortest edge and then select a sequence of edges of non-decreasing weights, checking at each stage that no cycle is completed ***(M1)***

choice	edge	weight
1	BG	1
2	AG	2
3	FG	3
4	BC	4
5	DE	5
6	AH	6
7	EG	7

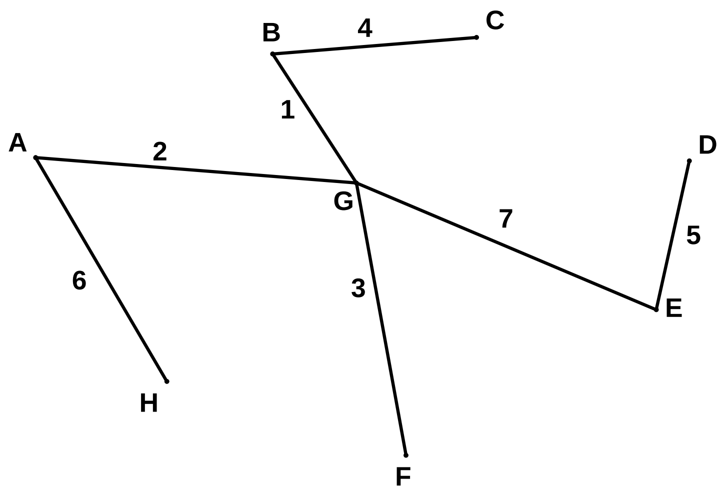
A1 (for choice 1)

A3

Note: ***A1*** for steps 2–4, ***A1*** for step 5 and ***A1*** for steps 6, 7.
Award marks only if it is clear that Kruskal's algorithm is being used.

[5 marks]

(b) weight $= 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$ ***A1***



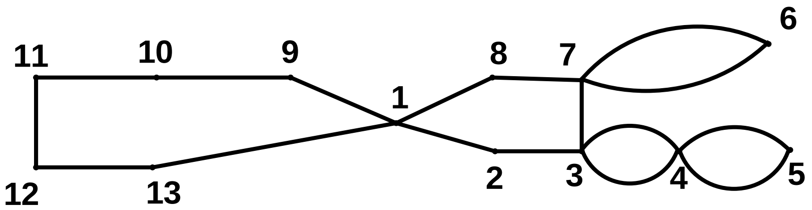
A1

Note: Award ***FT*** only if it is a spanning tree.

[2 marks]

Total [7 marks]

2. (a) (i)



(M1)A1

Note: Do not penalize candidates who include the entrance foyer.

(ii) the degrees of the vertices are 4, 2, 4, 4, 2, 2, 4, 2, 2, 2, 2, 2, 2 *A1*

(iii) the degree of all vertices is even and hence a Eulerian circuit exists, *A1*
hence it is possible to enter the museum through the foyer and visit each room 1–13 going through each internal doorway exactly once *AG*

Note: The connected graph condition is not required.

[4 marks]

(b) (i)

G

	A	**B**	**C**	**D**	**E**	**F**	total
A	0	2	0	2	0	0	4
B	2	0	1	1	0	1	5
C	0	1	0	1	2	1	5
D	2	1	1	0	2	0	6
E	0	0	2	2	0	2	6
F	0	1	1	0	2	0	4
						total	30

H

	P	**Q**	**R**	**S**	**T**	**U**	total
P	0	1	3	0	1	2	7
Q	1	0	1	3	2	0	7
R	3	1	0	2	1	3	10
S	0	3	2	0	2	0	7
T	1	2	1	2	0	1	7
U	2	0	3	0	1	0	6
						total	44

(M1)

graph *G* has 15 edges and graph *H* has 22 edges *A1A1*

(ii) the degree of every vertex is equal to the sum of the numbers in the corresponding row (or column) of the adjacency table
exactly two of the vertices of *G* have an odd degree (B and C) *A1*
H has four vertices with odd degree *A1*
G is the graph that has a Eulerian trail (and *H* does not) *R1*

(iii) neither graph has all vertices of even degree *R1*
therefore neither of them has a Eulerian circuit *AG*

[7 marks]

Total [11 marks]

3. (a) $10 \equiv 1(\text{mod}\,9) \Rightarrow 10^i \equiv 1(\text{mod}\,9)$, $i = 1, \ldots, n$ ***M1A1***

$\Rightarrow 10^i a_i \equiv a_i\,(\text{mod}\,9)$, $i = 1, \ldots, n$ ***M1***

> **Note:** Allow $i = 0$ but do not penalize its omission.

$\Rightarrow (10^n a_n + 10^{n-1}a_{n-1} + \ldots + a_0) \equiv (a_n + a_{n-1} + \ldots\ldots + a_0)(\text{mod}\,9)$ ***AG***

[3 marks]

(b) $4+7+6+x+2+1+2+y = 9k$, $k \in \mathbb{Z}$ ***(M1)***

$\Rightarrow (22+x+y) \equiv 0(\text{mod}\,9)$, $\Rightarrow (x+y) \equiv 5(\text{mod}\,9)$

$\Rightarrow x+y = 5$ or 14 ***A1***

if 5 divides a, then $y = 0$ or 5 ***M1***

so $y = 0 \Rightarrow x = 5$, $\left(ie\ (x,\ y) = (5,\ 0)\right)$ ***A1***

$y = 5 \Rightarrow x = 0$ or $x = 9$, $\left(ie\ (x,\ y) = (0,\ 5) \text{ or } (x,\ y) = (9,\ 5)\right)$ ***A1A1***

[6 marks]

(c) (i)

34390	**1**
3821	**5**
424	**1**
47	**2**
5	

(M1)A1

$b = (52151)_9$ ***AG***

(ii)

					5	2	1	5	1
				×	5	2	1	5	1
					5	2	1	5	1
			2	8	1	7	7	5	
			5	2	1	5	1		
	1	1	4	3	1	2			
2	8	1	7	7	5				
3	0	4	2	3	5	8	1	1	1

M1A3

[6 marks]

> **Note:** ***M1*** for attempt, ***A1*** for two correct lines of multiplication, ***A2*** for two correct lines of multiplication and a correct addition, ***A3*** for all correct.

Total [15 marks]

4. (a) *eg* the cycle $A \to B \to C \to D \to E \to A$ is Hamiltonian ***A1***
starting from any vertex there are four choices
from the next vertex there are three choices, *etc* … ***R1***
so the number of Hamiltonian cycles is $4!(=24)$ ***A1***

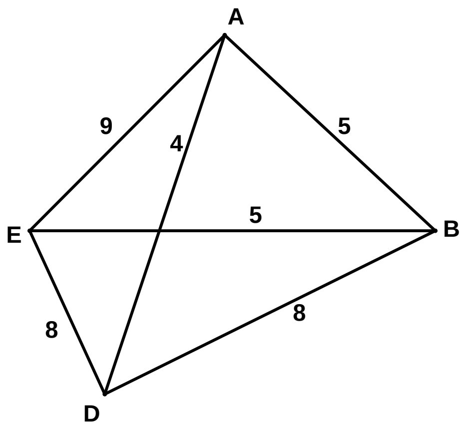
> **Note:** Allow 12 distinct cycles (direction not considered) or 60 (if different starting points count as distinct). In any case, just award the second ***A1*** if ***R1*** is awarded.

[3 marks]

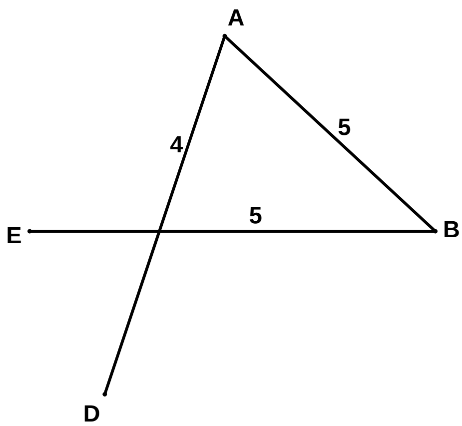
(b) total weight of any Hamiltonian cycles stated
eg $A \to B \to C \to D \to E \to A$ has weight $5+6+7+8+9=35$ ***A1***

[1 mark]

(c) consider the graph obtained from *G* after removing the vertex C



start (for instance) at A, using Prim's algorithm ***M1***
then D is the nearest vertex (add AD to the tree) ***A1***
next B is the nearest vertex (add AB to the tree) ***A1***
finally E is the nearest vertex (add BE to the tree) ***A1***
so a minimum spanning tree (of weight $4+5+5=14$) is



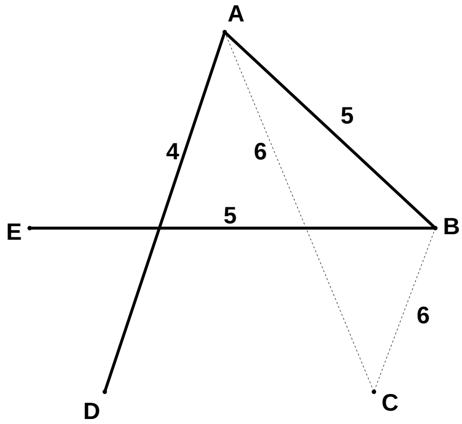
A1

[5 marks]

continued …

Question 4 continued

(d) a lower bound for the travelling salesman problem is then obtained by adding the weights of CA and CB to the weight of the minimum spanning tree ***(M1)***

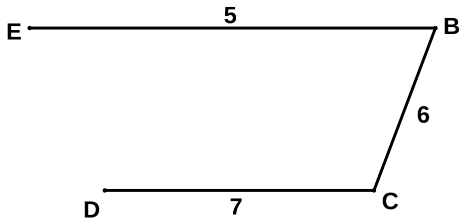


a lower bound is then $14+6+6=26$ ***A1***

[2 marks]

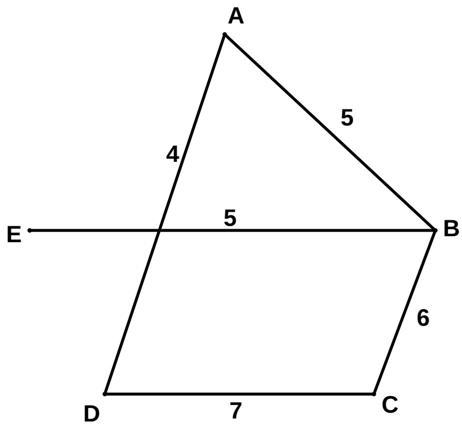
(e) **METHOD 1**

eg eliminating A from *G*, a minimum spanning tree of weight 18 is ***(M1)***



A1

adding AD and AB to the spanning tree gives a lower bound of $18+4+5=27>26$ ***A1***



so 26 is not the best lower bound ***AG***

Note: Candidates may delete other vertices and the lower bounds obtained are B-28, D-27 and E-28.

continued …

Question 4 continued

METHOD 2

there are 12 distinct cycles (ignoring direction) with the following lengths

Cycle	Length	
ABCDEA	35	***M1***
ABCEDA	33	
ABDCEA	39	
ABDECA	37	
ABECDA	31	
ABEDCA	31	
ACBDEA	37	
ACBEDA	**29**	
ACDBEA	35	
ACEBDA	33	
AEBCDA	31	
AECBDA	37	***A1***

as the optimal solution has length 29 ***A1***
26 is not the best possible lower bound ***AG***

Note: Allow answers where candidates list the 24 cycles obtained by allowing both directions.

[3 marks]

Total [14 marks]

5. (a) **METHOD 1**

$n^5 - n = \underbrace{n(n-1)(n+1)}_{\text{3 consecutive integers}}(n^2+1) \equiv 0 \pmod 6$ ***M1***

at least a factor is multiple of 3 and at least a factor is multiple of 2 ***R1***

$n^5 - n = n(n^4 - 1) \equiv 0 \pmod 5$ as $n^4 \equiv 1 \pmod 5$ by FLT ***R1***

therefore, as $(5, 6) = 1$, ***R1***

$n^5 - n \equiv 0\left(\bmod \underbrace{5\times 6}_{30}\right)$ ***A1***

ie 30 is a factor of $n^5 - n$ ***AG***

METHOD 2

let $\text{P}(n)$ be the proposition: $n^5 - n = 30\alpha$ for some $\alpha \in \mathbb{Z}$

$0^5 - 0 = 30\times 0$, so $\text{P}(0)$ is true ***A1***

assume $\text{P}(k)$ is true for some k and consider $\text{P}(k+1)$

$(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$ ***M1***

$= (k^5 - k) + 5k\left(\underbrace{k^3 + 3k^2 + 3k + 1}_{(k+1)^3} - (k^2 + k)\right)$

$= (k^5 - k) + 5k\left((k+1)^3 - k(k+1)\right)$

$= 30\alpha + \underbrace{5k(k+1)\left(\underbrace{k^2 + k + 1}_{k(k+1)+1}\right)}_{\text{multiple of 6}}$ ***M1***

$= 30\alpha + 30\beta$ ***A1***

as $\text{P}(0)$ is true and $\text{P}(k)$ true implies $\text{P}(k+1)$ true, by PMI $\text{P}(n)$ is true for all values $n \in \mathbb{N}$ ***R1***

Note:	Award the first ***M1*** only if the correct induction procedure is followed and the correct first line is seen.

Note:	Award ***R1*** only if both ***M*** marks have been awarded.

METHOD 3

$n^5 - n = n(n^4 - 1)$ ***M1***

$= n(n^2 - 1)(n^2 + 1)$ ***A1***

$= (n-1)n(n+1)(n^2 - 4 + 5)$ ***R1***

$= (n-2)(n-1)n(n+1)(n+2) + 5(n-1)n(n+1)$ ***A1***

each term is multiple of 2, 3 and 5 ***R1***

therefore is divisible by 30 ***AG***

[5 marks]

continued …

Question 5 continued

(b) (i) **METHOD 1**

case 1: $m = 0$ and $3^{3^0} \equiv 3 \bmod 4$ is true **A1**
case 2: $m > 0$
let $N = 3^m \geq 3$ and consider the binomial expansion **M1**

$$3^N = (1+2)^N = \sum_{k=0}^{N} \binom{N}{k} 2^k = 1 + 2N + \underbrace{\sum_{k=2}^{N} \binom{N}{k} 2^k}_{\equiv 0 (\bmod 4)} \equiv 1 + 2N \ (\bmod 4)$$ **A1**

as $\underbrace{3^m}_{N} \equiv (-1)^m (\bmod 4) \Rightarrow 1 + 2N \equiv 1 + 2(-1)^m (\bmod 4)$ **R1**

therefore $\underbrace{3^{3^m}}_{3^N} \equiv 1 + 2(-1)^m (\bmod 4) \Rightarrow \begin{cases} \underbrace{3^{3^m}}_{3^N} \equiv \underbrace{1+2}_{3} \ (\text{mod } 4) \text{ for } m \text{ even} \\ \underbrace{3^{3^m}}_{3^N} \equiv \underbrace{1-2}_{-1 \equiv 3 (\bmod 4)} \ (\text{mod } 4) \text{ for } m \text{ odd} \end{cases}$ **R1**

which proves that $3^{3^m} \equiv 3 (\bmod 4)$ for any $m \in \mathbb{N}$ **AG**

METHOD 2

let $\mathrm{P}(n)$ be the proposition: $3^{3^n} - 3 \equiv 0 (\text{mod } 4, \text{ or } 24)$

$3^{3^0} - 3 = 3 - 3 \equiv 0 (\text{mod } 4 \text{ or } 24)$, so $\mathrm{P}(0)$ is true **A1**
assume $\mathrm{P}(k)$ is true for some k **M1**

consider $3^{3^{k+1}} - 3 = 3^{3^k \times 3} - 3$ **M1**
$= (3 + 24r)^3 - 3$
$\equiv 27 + 24t - 3$ **R1**
$\equiv 0$ (mod 4 or 24)

as $\mathrm{P}(0)$ is true and $\mathrm{P}(k)$ true implies $\mathrm{P}(k+1)$ true, by PMI $\mathrm{P}(n)$ is true for all values $n \in \mathbb{N}$ **R1**

METHOD 3

$3^{3^m} - 3 = 3(3^{3^m - 1} - 1)$ **M1A1**
$= 3(3^{2k} - 1)$ **R1**
$= 3(9^k - 1)$
$= 3\underbrace{\left((8+1)^k - 1\right)}_{\text{multiple of } 8}$ **R1**
$\equiv 0 (\bmod 24)$ **A1**

which proves that $3^{3^m} \equiv 3 (\bmod 4)$ for any $m \in \mathbb{N}$ **AG**

continued …

Question 5 continued

(ii) for $m \in \mathbb{N}$, $3^{3^m} \equiv 3(\bmod 4)$ and, as $2^{2^n} \equiv 0(\bmod 4)$ and $5^2 \equiv 1(\bmod 4)$ then

$2^{2^n} + 5^2 \equiv 1(\bmod 4)$ for $n \in \mathbb{Z}^+$

there is no solution to $3^{3^m} = 2^{2^n} + 5^2$ for pairs $(m, n) \in \mathbb{N} \times \mathbb{Z}^+$ ***R1***

when $n = 0$, we have $3^{3^m} = 2^{2^0} + 5^2 \Rightarrow 3^{3^m} = 27 \Rightarrow m = 1$ ***M1***

therefore $(m, n) = (1, 0)$ ***A1***

is the only pair of non-negative integers that satisfies the equation ***AG***

[8 marks]

Total [13 marks]