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MATHEMATICS
HIGHER LEVEL
PAPER 3 – SETS, RELATIONS AND GROUPS

Thursday 8 November 2012 (morning)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics SL*** *information booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 19]*

All of the relations in this question are defined on $\mathbb{Z} \setminus \{0\}$.

(a) Decide, giving a proof or a counter-example, whether $xRy \Leftrightarrow x + y > 7$ is

(i) reflexive;

(ii) symmetric;

(iii) transitive. *[4 marks]*

(b) Decide, giving a proof or a counter-example, whether $xRy \Leftrightarrow -2 < x - y < 2$ is

(i) reflexive;

(ii) symmetric;

(iii) transitive. *[4 marks]*

(c) Decide, giving a proof or a counter-example, whether $xRy \Leftrightarrow xy > 0$ is

(i) reflexive;

(ii) symmetric;

(iii) transitive. *[4 marks]*

(d) Decide, giving a proof or a counter-example, whether $xRy \Leftrightarrow \frac{x}{y} \in \mathbb{Z}$ is

(i) reflexive;

(ii) symmetric;

(iii) transitive. *[4 marks]*

(e) One of the relations from parts (a), (b), (c) and (d) is an equivalence relation. For this relation, state what the equivalence classes are. *[3 marks]*

2. *[Maximum mark: 9]*

Let $\boldsymbol{A}$ be the set of 2×1 matrices defined as follows: $\boldsymbol{A}=\left\{\begin{pmatrix}x\\y\end{pmatrix}\middle|\, x,\ y\in\mathbb{R}\right\}$. A function f is defined from $\boldsymbol{A}$ to $\boldsymbol{A}$ by $f\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$.

(a) Evaluate $f\left(\begin{pmatrix}5\\6\end{pmatrix}\right)$. *[1 mark]*

(b) Prove that f is an injection. *[2 marks]*

(c) Prove that f is a surjection. *[2 marks]*

(d) Find $f^{-1}\left(\begin{pmatrix}x\\y\end{pmatrix}\right)$. *[2 marks]*

Another function g is defined from $\boldsymbol{A}$ to $\boldsymbol{A}$ by $g\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}1&2\\3&6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$.

(e) Is g a bijection? Justify your answer. *[2 marks]*

3. *[Maximum mark: 15]*

Let $A = \{a, b\}$.

(a) Write down all four subsets of A. *[1 mark]*

Let the set of all these subsets be denoted by $P(A)$. The binary operation symmetric difference, Δ, is defined on $P(A)$ by $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$ where $X, Y \in P(A)$.

(b) Construct the Cayley table for $P(A)$ under Δ. *[3 marks]*

(c) Prove that $\{P(A), \Delta\}$ is a group. You are allowed to assume that Δ is associative. *[3 marks]*

Let $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ and $+_4$ denote addition modulo 4.

(d) Is $\{P(A), \Delta\}$ isomorphic to $\{\mathbb{Z}_4, +_4\}$? Justify your answer. *[2 marks]*

Let S be any non-empty set. Let $P(S)$ be the set of all subsets of S. For the following parts, you are allowed to assume that Δ, $\cup$ and $\cap$ are associative.

(e) (i) State the identity element for $\{P(S), \Delta\}$.

(ii) Write down X^{-1} for $X \in P(S)$.

(iii) Hence prove that $\{P(S), \Delta\}$ is a group. *[4 marks]*

(f) Explain why $\{P(S), \cup\}$ is not a group. *[1 mark]*

(g) Explain why $\{P(S), \cap\}$ is not a group. *[1 mark]*

4. *[Maximum mark: 17]*

Let c be a positive, real constant. Let G be the set $\{x \in \mathbb{R} \mid -c < x < c\}$. The binary operation $*$ is defined on the set G by $x * y = \dfrac{x+y}{1+\dfrac{xy}{c^2}}$.

(a) Simplify $\dfrac{c}{2} * \dfrac{3c}{4}$. *[2 marks]*

(b) State the identity element for G under $*$. *[1 mark]*

(c) For $x \in G$ find an expression for x^{-1} (the inverse of x under $*$). *[1 mark]*

(d) Show that the binary operation $*$ is commutative on G. *[2 marks]*

(e) Show that the binary operation $*$ is associative on G. *[4 marks]*

(f) (i) If $x, y \in G$ explain why $(c-x)(c-y) > 0$.

(ii) Hence show that $x + y < c + \dfrac{xy}{c}$. *[2 marks]*

You are also told that $-c - \dfrac{xy}{c} < x + y$.

(g) Show that G is closed under $*$. *[2 marks]*

(h) Explain why $\{G, *\}$ is an Abelian group. *[2 marks]*

(i) State what happens to the group $\{G, *\}$ as $c \to \infty$. *[1 mark]*