N12/5/MATHL/HP1/ENG/TZ0/XX/M

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MARKSCHEME

November 2012

MATHEMATICS

Higher Level

Paper 1

17 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to scoris instructions and the document "**Mathematics HL: Guidance for e-marking Nov 2012**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)****.*

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT****.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

SECTION A

1. $\sin\alpha = \sqrt{1-\left(-\frac{3}{4}\right)^2} = \frac{\sqrt{7}}{4}$ ***(M1)A1***

attempt to use double angle formula ***M1***

$\sin 2\alpha = 2\frac{\sqrt{7}}{4}\left(-\frac{3}{4}\right) = -\frac{3\sqrt{7}}{8}$ ***A1***

Note: $\frac{\sqrt{7}}{4}$ seen would normally be awarded ***M1A1***.

[4 marks]

2. $\left(\frac{x}{y}-\frac{y}{x}\right)^4 = \left(\frac{x}{y}\right)^4 + 4\left(\frac{x}{y}\right)^3\left(-\frac{y}{x}\right) + 6\left(\frac{x}{y}\right)^2\left(-\frac{y}{x}\right)^2 + 4\left(\frac{x}{y}\right)\left(-\frac{y}{x}\right)^3 + \left(-\frac{y}{x}\right)^4$ ***(M1)(A1)***

Note: Award ***M1*** for attempt to expand and ***A1*** for correct unsimplified expansion.

$= \frac{x^4}{y^4} - 4\frac{x^2}{y^2} + 6 - 4\frac{y^2}{x^2} + \frac{y^4}{x^4} \quad \left(= \frac{x^8 - 4x^6y^2 + 6x^4y^4 - 4x^2y^6 + y^8}{x^4y^4}\right)$ ***A1A1***

Note: Award ***A1*** for powers, ***A1*** for coefficients and signs.

Note: Final two ***A*** marks are independent of first ***A*** mark.

[4 marks]

3. (a) **METHOD 1**

$f(x) = (x+1)(x-1)(x-2)$ ***M1***

$= x^3 - 2x^2 - x + 2$ ***A1A1A1***

$a = -2$, $b = -1$ and $c = 2$

METHOD 2

from the graph or using $f(0) = 2$

$c = 2$ ***A1***

setting up linear equations using $f(1) = 0$ and $f(-1) = 0$ (or $f(2) = 0$) ***M1***

obtain $a = -2$, $b = -1$ ***A1A1***

[4 marks]

(b) (i) $(1, 0)$, $(3, 0)$ and $(4, 0)$ ***A1***

(ii) $g(0)$ occurs at $3f(-2)$ ***(M1)***

$= -36$ ***A1***

[3 marks]

Total [7 marks]

4. (a) $f'(x) = (\ln x)^2 + \frac{2x\ln x}{x}\left(= (\ln x)^2 + 2\ln x = \ln x(\ln x + 2)\right)$ ***M1A1***

$f'(x) = 0$ ($\Rightarrow x = 1,\ x = e^{-2}$) ***M1***

Note: Award ***M1*** for an attempt to solve $f'(x) = 0$.

$A(e^{-2}, 4e^{-2})$ **and** $B(1, 0)$ ***A1A1***

Note: The final ***A1*** is independent of prior working.

[5 marks]

(b) $f''(x) = \frac{2}{x}(\ln x + 1)$ ***A1***

$f''(x) = 0\ \left(\Rightarrow x = e^{-1}\right)$ ***(M1)***

inflexion point (e^{-1}, e^{-1}) ***A1***

Note: ***M1*** for attempt to solve $f''(x) = 0$.

[3 marks]

Total [8 marks]

5. (a) 0 ***A1***

[1 mark]

(b) $\int_0^1 f(x)\,dx = 1$ ***(M1)***

$\Rightarrow a = \frac{1}{\int_0^1 e^{-x}dx}$

$\Rightarrow a = \frac{1}{\left[-e^{-x}\right]_0^1}$ ***A1***

$\Rightarrow a = \frac{e}{e-1}$ (or equivalent) ***A1***

Note: Award first ***A1*** for correct integration of $\int e^{-x}dx$. This ***A1*** is independent of previous ***M*** mark.

[3 marks]

(c) $\mathrm{E}(X) = \int_0^1 xf(x)\,dx\left(= a\int_0^1 xe^{-x}\,dx\right)$ ***M1***

attempt to integrate by parts ***M1***

$= a\left[-xe^{-x} - e^{-x}\right]_0^1$ ***(A1)***

$= a\left(\frac{e-2}{e}\right)$

$= \frac{e-2}{e-1}$ (or equivalent) ***A1***

[4 marks]

Total [8 marks]

6. **METHOD 1**

(a) $\det\begin{pmatrix}1 & 3 & a-1\\ 2 & 2 & a-2\\ 3 & 1 & a-3\end{pmatrix}$ ***M1***

$=1\big(2(a-3)-(a-2)\big)-3\big(2(a-3)-3(a-2)\big)+(a-1)(2-6)$

(or equivalent) ***A1***

$=0$ (therefore there is no unique solution) ***A1***

[3 marks]

(b) $\left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 2 & 2 & a-2 & 1\\ 3 & 1 & a-3 & b\end{array}\right) : \left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 0 & -4 & -a & -1\\ 0 & -8 & -2a & b-3\end{array}\right)$ ***M1A1***

$: \left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 0 & -4 & -a & -1\\ 0 & 0 & 0 & b-1\end{array}\right)$ ***A1***

$b=1$ ***A1*** ***N2***

Note: Award ***M1*** for an attempt to use row operations.

[4 marks]

METHOD 2

(a) $\left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 2 & 2 & a-2 & 1\\ 3 & 1 & a-3 & b\end{array}\right) : \left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 0 & -4 & -a & -1\\ 0 & -8 & -2a & b-3\end{array}\right)$ ***M1A1***

$: \left(\begin{array}{ccc|c}1 & 3 & a-1 & 1\\ 0 & -4 & -a & -1\\ 0 & 0 & 0 & b-1\end{array}\right)$ (and 3 zeros imply no unique solution) ***A1***

[3 marks]

(b) $b=1$ ***A4***

Note: Award ***A4*** only if "$b-1$" seen in (a).

[4 marks]

Total [7 marks]

7. (a) attempt to apply cosine rule *M1*

$4^2 = 6^2 + QR^2 - 2 \cdot QR \cdot 6\cos 30^\circ$ $\left(\textbf{or } QR^2 - 6\sqrt{3}\ QR + 20 = 0\right)$ *A1*

$QR = 3\sqrt{3} + \sqrt{7}$ or $QR = 3\sqrt{3} - \sqrt{7}$ *A1A1*

[4 marks]

(b) **METHOD 1**

$k \geq 6$ *A1*

$k = 6\sin 30^\circ = 3$ *M1A1*

Note: The ***M1*** in (b) is for recognizing the right-angled triangle case.

METHOD 2

$k \geq 6$ *A1*

use of discriminant: $108 - 4\left(36 - k^2\right) = 0$ *M1*

$k = 3$ *A1*

Note: $k = \pm 3$ is ***M1A0.***

[3 marks]

Total [7 marks]

8. (a) attempt to differentiate implicitly *M1*

$2x + \cos y \dfrac{dy}{dx} - y - x\dfrac{dy}{dx} = 0$ *A1A1*

Note: ***A1*** for differentiating x^2 and $\sin y$; ***A1*** for differentiating xy.

substitute x and y by π *M1*

$2\pi - \dfrac{dy}{dx} - \pi - \pi\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{\pi}{1+\pi}$ *M1A1*

Note: ***M1*** for attempt to make dy/dx the subject. This could be seen earlier.

[6 marks]

(b) $\theta = \dfrac{\pi}{4} - \arctan\dfrac{\pi}{1+\pi}$ (or seen the other way) *M1*

$\tan\theta = \tan\left(\dfrac{\pi}{4} - \arctan\dfrac{\pi}{1+\pi}\right) = \dfrac{1 - \dfrac{\pi}{1+\pi}}{1 + \dfrac{\pi}{1+\pi}}$ *M1A1*

$\tan\theta = \dfrac{1}{1+2\pi}$ *AG*

[3 marks]

Total [9 marks]

9. **METHOD 1**

(a) $9t_A = 7 - 4t_B$ and
$3 - 6t_A = -6 + 7t_B$ ***M1A1***

solve simultaneously

$t_A = \frac{1}{3},\ t_B = 1$ ***A1***

Note: Only need to see one time for the ***A1***.

therefore meet at $(3, 1)$ ***A1***

[4 marks]

(b) boats do not collide because the two times $\left(t_A = \frac{1}{3},\ t_B = 1\right)$ ***(A1)***
are different ***R1***

[2 marks]

Total [6 marks]

METHOD 2

(a) path of A is a straight line: $y = -\frac{2}{3}x + 3$ ***M1A1***

Note: Award ***M1*** for an attempt at simultaneous equations.

path of B is a straight line: $y = -\frac{7}{4}x + \frac{25}{4}$ ***A1***

$-\frac{2}{3}x + 3 = -\frac{7}{4}x + \frac{25}{4}\ (\Rightarrow x = 3)$
so the common point is $(3, 1)$ ***A1***

[4 marks]

(b) for boat A, $9t = 3 \Rightarrow t = \frac{1}{3}$ and for boat B, $7 - 4t = 3 \Rightarrow t = 1$ ***A1***

times are different so boats do not collide ***R1AG***

[2 marks]

Total [6 marks]

SECTION B

10. (a) (i) $z_1 = 2\sqrt{3}\,\text{cis}\frac{3\pi}{2} \Rightarrow z_1 = -2\sqrt{3}\text{i}$ ***A1***

(ii) $z_1 + z_2 = -2\sqrt{3}\text{i} - 1 + \sqrt{3}\text{i} = -1 - \sqrt{3}\text{i}$ ***A1***

$(z_1 + z_2)^* = -1 + \sqrt{3}\text{i}$ ***A1***

[3 marks]

(b) (i) $|z_2| = 2$

$\tan\theta = -\sqrt{3}$ ***(M1)***

z_2 lies on the second quadrant

$\theta = \arg z_2 = \frac{2\pi}{3}$

$z_2 = 2\text{cis}\frac{2\pi}{3}$ ***A1A1***

(ii) attempt to use De Moivre's theorem ***M1***

$z = \sqrt[3]{2}\,\text{cis}\frac{\frac{2\pi}{3} + 2k\pi}{3}, \ k = 0, 1 \text{ and } 2$

$z = \sqrt[3]{2}\,\text{cis}\frac{2\pi}{9}, \ \sqrt[3]{2}\,\text{cis}\frac{8\pi}{9}, \ \sqrt[3]{2}\,\text{cis}\frac{14\pi}{9}\left(= \sqrt[3]{2}\,\text{cis}\left(\frac{-4\pi}{9}\right)\right)$ ***A1A1***

Note: Award ***A1*** for modulus, ***A1*** for arguments.

Note: Allow equivalent forms for z.

[6 marks]

(c) (i) **METHOD 1**

$z^2 = \left(1 - 1 + \sqrt{3}\text{i}\right)^2 = -3\left(\Rightarrow z = \pm\sqrt{3}\text{i}\right)$ ***M1***

$z = \sqrt{3}\,\text{cis}\frac{\pi}{2}$ or $z_1 = \sqrt{3}\,\text{cis}\frac{3\pi}{2}\left(= \sqrt{3}\,\text{cis}\left(\frac{-\pi}{2}\right)\right)$ ***A1A1***

so $r = \sqrt{3}$ and $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}\left(= \frac{-\pi}{2}\right)$

Note: Accept $r\text{cis}(\theta)$ form.

METHOD 2

$z^2 = \left(1 - 1 + \sqrt{3}\text{i}\right)^2 = -3 \Rightarrow z^2 = 3\text{cis}\left((2n+1)\pi\right)$ ***M1***

$r^2 = 3 \Rightarrow r = \sqrt{3}$ ***A1***

$2\theta = (2n+1)\pi \Rightarrow \theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ (as $0 \le \theta < 2\pi$) ***A1***

Note: Accept $r\text{cis}(\theta)$ form.

continued ...

Question 10 continued

(ii) **METHOD 1**

$z = -\dfrac{1}{2cis\dfrac{2\pi}{3}} \Rightarrow z = \dfrac{\text{cis}\pi}{2\text{cis}\dfrac{2\pi}{3}}$ ***M1***

$\Rightarrow z = \dfrac{1}{2}\text{cis}\dfrac{\pi}{3}$

so $r = \dfrac{1}{2}$ and $\theta = \dfrac{\pi}{3}$ ***A1A1***

METHOD 2

$z_1 = -\dfrac{1}{-1+\sqrt{3}\text{i}} \Rightarrow z_1 = -\dfrac{-1-\sqrt{3}\text{i}}{\left(-1+\sqrt{3}\text{i}\right)\left(-1-\sqrt{3}\text{i}\right)}$ ***M1***

$z = \dfrac{1+\sqrt{3}\text{i}}{4} \Rightarrow z = \dfrac{1}{2}\text{cis}\dfrac{\pi}{3}$

so $r = \dfrac{1}{2}$ and $\theta = \dfrac{\pi}{3}$ ***A1A1***

[6 marks]

(d) $\dfrac{z_1}{z_2} = \sqrt{3}\,\text{cis}\dfrac{5\pi}{6}$ (***A1)***

$\left(\dfrac{z_1}{z_2}\right)^n = \sqrt{3}^n\text{cis}\dfrac{5n\pi}{6}$ ***A1***

equating imaginary part to zero and attempting to solve ***M1***

obtain $n = 12$ ***A1***

Note: Working which only includes the argument is valid.

[4 marks]

Total [19 marks]

11. (a) $\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \boldsymbol{A}^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$ ***A1***

$\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow 3\boldsymbol{A} = \begin{pmatrix} 6 & 3 \\ 3 & 3 \end{pmatrix}$ ***(A1)***

$(\boldsymbol{A}^2 - 3\boldsymbol{A})^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^2$ ***A1***

$= \boldsymbol{I}$ ***AG***

[3 marks]

(b) (i) $\begin{pmatrix} a & a-1 \\ b & b \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \Rightarrow \begin{cases} a + 3a - 3 = 0 \\ b + 3b = -2 \end{cases}$ ***M1(A1)***

$a = \frac{3}{4}$ and $b = -\frac{1}{2}$ ***A1***

(ii) **METHOD 1**

$\boldsymbol{A}^{-1}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \Rightarrow \boldsymbol{A}\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ ***M1***

$\begin{pmatrix} a & a-1 \\ b & b \end{pmatrix}\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \begin{cases} -2a + 2 = 1 \\ -2b = 3 \end{cases}$ ***A1***

$a = \frac{1}{2}$ and $b = -\frac{3}{2}$ ***A1***

METHOD 2

serious attempt at finding $\boldsymbol{A}^{-1}$ ***M1***

$\left(\boldsymbol{A}^{-1} = \frac{1}{b}\begin{pmatrix} b & 1-a \\ -b & a \end{pmatrix}\right)$

obtain $b + 3 - 3a = 0$ and $-b + 3a = -2b$ or equivalent linear equations ***A1***

$a = \frac{1}{2}$ and $b = -\frac{3}{2}$ ***A1***

[6 marks]

continued …

Question 11 continued

(c) (i) **METHOD 1**

$$\underbrace{\begin{pmatrix} a & a-1 \\ b & b \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ ***M1***

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & a-1 \\ b & b \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ ***(A1)***

$$\boldsymbol{A}^{-1} = \frac{1}{b}\begin{pmatrix} b & -a+1 \\ -b & a \end{pmatrix}$$ ***A1***

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-a+1}{b} \\ \frac{a}{b} \end{pmatrix}$$ ***A1***

METHOD 2

$ax + (a-1)y = 0$ and $bx + by = 1$ ***A1***

attempt to solve ***M1***

obtain $\left(\frac{1-a}{b}, \frac{a}{b}\right)$ ***A1A1***

(ii) gradient of l_1 is $\frac{-a}{a-1}$ and gradient of l_2 is -1 ***A1A1***

the lines are perpendicular $\Rightarrow \frac{-a}{a-1} = 1 \Rightarrow a = \frac{1}{2}$ ***M1A1***

so they intersect at $\left(\frac{1}{2b}, \frac{1}{2b}\right)$ ***A1***

[9 marks]

Total [18 marks]

12. (a) $(f \circ f)(x) = f\left(\dfrac{x}{2-x}\right) = \dfrac{\frac{x}{2-x}}{2-\frac{x}{2-x}}$ *M1A1*

$(f \circ f)(x) = \dfrac{x}{4-3x}$ *A1*

[3 marks]

(b) $P(n): \underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(x) = F_n(x)$

$P(1):\ f(x) = F_1(x)$

$LHS = f(x) = \dfrac{x}{2-x}$ and $RHS = F_1(x) = \dfrac{x}{2^1 - (2^1 - 1)x} = \dfrac{x}{2-x}$ *A1A1*

$\therefore P(1)$ true

assume that $P(k)$ is true, *i.e.*, $\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k times}}(x) = F_k(x)$ *M1*

consider $P(k+1)$

EITHER

$\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k+1 times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{\text{k times}}\right)(x) = f\left(F_k(x)\right)$ *(M1)*

$= f\left(\dfrac{x}{2^k - (2^k - 1)x}\right) = \dfrac{\frac{x}{2^k - (2^k-1)x}}{2 - \frac{x}{2^k - (2^k-1)x}}$ *A1*

$= \dfrac{x}{2\left(2^k - (2^k - 1)x\right) - x} = \dfrac{x}{2^{k+1} - (2^{k+1} - 2)x - x}$ *A1*

OR

$\underbrace{(f \circ f \circ \ldots \circ f)}_{\text{k+1 times}}(x) = \left(f \circ \underbrace{f \circ f \circ \ldots \circ f}_{\text{k times}}\right)(x) = F_k(f(x))$ *(M1)*

$= F_k\left(\dfrac{x}{2-x}\right) = \dfrac{\frac{x}{2-x}}{2^k - (2^k - 1)\frac{x}{2-x}}$ *A1*

$= \dfrac{x}{2^{k+1} - 2^k x - 2^k x + x}$ *A1*

THEN

$= \dfrac{x}{2^{k+1} - (2^{k+1} - 1)x} = F_{k+1}(x)$ *A1*

$P(k)$ true implies $P(k+1)$ true, $P(1)$ true so $P(n)$ true for all $n \in \mathbb{Z}^+$ *R1*

[8 marks]

continued ...

Question 12 continued

(c) **METHOD 1**

$x = \dfrac{y}{2^n - (2^n - 1)y} \Rightarrow 2^n x - (2^n - 1)xy = y$ ***M1A1***

$\Rightarrow 2^n x = \left((2^n - 1)x + 1\right)y \Rightarrow y = \dfrac{2^n x}{(2^n - 1)x + 1}$ ***A1***

$F_n^{-1}(x) = \dfrac{2^n x}{(2^n - 1)x + 1}$ ***A1***

$F_n^{-1}(x) = \dfrac{x}{\frac{2^n - 1}{2^n}x + \frac{1}{2^n}}$ ***M1***

$F_n^{-1}(x) = \dfrac{x}{(1 - 2^{-n})x + 2^{-n}}$ ***A1***

$F_n^{-1}(x) = \dfrac{x}{2^{-n} - (2^{-n} - 1)x}$ ***AG***

METHOD 2

attempt $F_{-n}\left(F_n(x)\right)$ ***M1***

$= F_{-n}\left(\dfrac{x}{2^n - (2^n - 1)x}\right) = \dfrac{\dfrac{x}{2^n - (2^n - 1)x}}{2^{-n} - (2^{-n} - 1)\dfrac{x}{2^n - (2^n - 1)x}}$ ***A1A1***

$= \dfrac{x}{2^{-n}(2^n - (2^n - 1)x) - (2^{-n} - 1)x}$ ***A1A1***

Note: Award ***A1*** marks for numerators and denominators.

$= \dfrac{x}{1} = x$ ***A1AG***

METHOD 3

attempt $F_n\left(F_{-n}(x)\right)$ ***M1***

$= F_n\left(\dfrac{x}{2^{-n} - (2^{-n} - 1)x}\right) = \dfrac{\dfrac{x}{2^{-n} - (2^{-n} - 1)x}}{2^n - (2^n - 1)\dfrac{x}{2^{-n} - (2^{-n} - 1)x}}$ ***A1A1***

$= \dfrac{x}{2^n\left(2^{-n} - (2^{-n} - 1)x\right) - (2^n - 1)x}$ ***A1A1***

Note: Award ***A1*** marks for numerators and denominators.

$= \dfrac{x}{1} = x$ ***A1AG***

[6 marks]

continued ...

Question 12 continued

(d) (i) $F_n(0)=0,\ F_n(1)=1$ ***A1***

(ii) **METHOD 1**

$2^n-(2^n-1)x-1=(2^n-1)(1-x)$ ***(M1)***

>0 if $0<x<1$ and $n\in\mathbb{Z}^+$ ***A1***

so $2^n-(2^n-1)x>1$ and $F_n(x)=\dfrac{x}{2^n-(2^n-1)x}<\dfrac{x}{1}\ (<x)$ ***R1***

$F_n(x)=\dfrac{x}{2^n-(2^n-1)x}<x$ for $0<x<1$ and $n\in\mathbb{Z}^+$ ***AG***

METHOD 2

$\dfrac{x}{2^n-(2^n-1)x}<x \Leftrightarrow 2^n-(2^n-1)x>1$ ***(M1)***

$\Leftrightarrow (2^n-1)x<2^n-1$ ***A1***

$\Leftrightarrow x<\dfrac{2^n-1}{2^n-1}=1$ true in the interval $]0,1[$ ***R1***

(iii) $B_n=2\left(A_n-\dfrac{1}{2}\right)\ (=2A_n-1)$ ***(M1)A1***

[6 marks]

Total [23 marks]