M12/5/MATHL/HP2/ENG/TZ2/XX/M

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MARKSCHEME

May 2012

MATHEMATICS

Higher Level

Paper 2

20 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to scoris instructions and the document **"Mathematics HL: Guidance for e-marking May 2012"**. It is essential that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the 'must be seen' marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features (e.g. TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

SECTION A

1. (a) $S_n = \frac{n}{2}[2a + (n-1)d]$

$212 = \frac{16}{2}(2a + 15d) \quad (= 16a + 120d)$ *A1*

n^{th} term is $a + (n-1)d$

$8 = a + 4d$ *A1*

solving simultaneously: *(M1)*

$d = 1.5,\ a = 2$ *A1*

[4 marks]

(b) $\frac{n}{2}[4 + 1.5(n-1)] > 600$ *(M1)*

$\Rightarrow 3n^2 + 5n - 2400 > 0$ *(A1)*

$\Rightarrow n > 27.4\ldots,\ (n < -29.1\ldots)$

Note: Do not penalize improper use of inequalities.

$\Rightarrow n = 28$ *A1*

[3 marks]

Total [7 marks]

2. (a) $\text{E}(X) = np$

$\Rightarrow 10 = 30p$

$\Rightarrow p = \frac{1}{3}$ *A1*

[1 mark]

(b) $\text{P}(X = 10) = \binom{30}{10}\left(\frac{1}{3}\right)^{10}\left(\frac{2}{3}\right)^{20} = 0.153$ *(M1)A1*

[2 marks]

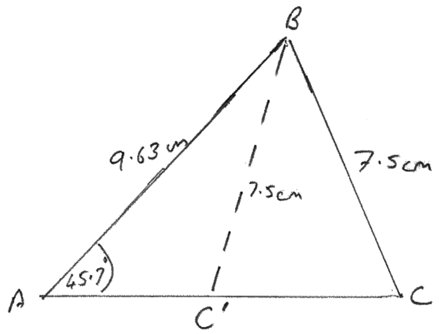
(c) $\text{P}(X \ge 15) = 1 - \text{P}(X \le 14)$ *(M1)*

$= 1 - 0.9565\ldots = 0.0435$ *A1*

[2 marks]

Total [5 marks]

3. (a)



A2

Note:	Accept 2 separate triangles. The diagram(s) should show that one triangle has an acute angle and the other triangle has an obtuse angle. The values 9.63, 7.5 and 45.7 and/or the letters, A, B C′ and C should be correctly marked on the diagram(s).

[2 marks]

(b) **METHOD 1**

$\frac{\sin 45.7}{7.5} = \frac{\sin C}{9.63}$ ***M1***

$\Rightarrow \hat{C} = 66.77\ldots^\circ, 113.2\ldots^\circ$ ***(A1)(A1)***

$\Rightarrow \hat{B} = 67.52\ldots^\circ, 21.07\ldots^\circ$ ***(A1)***

$\frac{b}{\sin B} = \frac{7.5}{\sin 45.7} \Rightarrow b = 9.68(\text{cm}), b = 3.77(\text{cm})$ ***A1A1***

Note:	If only the acute value of $\hat{C}$ is found, award ***M1(A1)(A0)(A0)A1A0***.

METHOD 2

$7.5^2 = 9.63^2 + b^2 - 2 \times 9.63 \times b \cos 45.7^\circ$ ***M1A1***

$b^2 - 13.45\ldots b + 36.48\ldots = 0$

$b = \frac{13.45\ldots \pm \sqrt{13.45\ldots^2 - 4 \times 36.48\ldots}}{2}$ ***(M1)(A1)***

$\text{AC} = 9.68(\text{cm}), \text{AC} = 3.77(\text{cm})$ ***A1A1***

[6 marks]

Total [8 marks]

4. (a) number of arrangements of boys is $15!$ and number of arrangements of girls is $10!$ ***(A1)***

total number of arrangements is $15! \times 10! \times 2 \, (= 9.49 \times 10^{18})$ ***M1A1***

Note: If 2 is omitted, award ***(A1)M1A0***.

[3 marks]

(b) number of ways of choosing two boys is $\binom{15}{2}$ and the number of ways of choosing three girls is $\binom{10}{3}$ ***(A1)***

number of ways of choosing two boys and three girls is $\binom{15}{2} \times \binom{10}{3} = 12600$ ***M1A1***

[3 marks]

Total [6 marks]

5. (a) $\mathrm{P}(X = 5) = \mathrm{P}(X = 3) + \mathrm{P}(X = 4)$

$\frac{\mathrm{e}^{-m} m^5}{5!} = \frac{\mathrm{e}^{-m} m^3}{3!} + \frac{\mathrm{e}^{-m} m^4}{4!}$ ***M1(A1)***

$m^2 - 5m - 20 = 0$

$\Rightarrow m = \frac{5 + \sqrt{105}}{2} = (7.62)$ ***A1***

[3 marks]

(b) $\mathrm{P}(X > 2) = 1 - \mathrm{P}(X \le 2)$ ***(M1)***

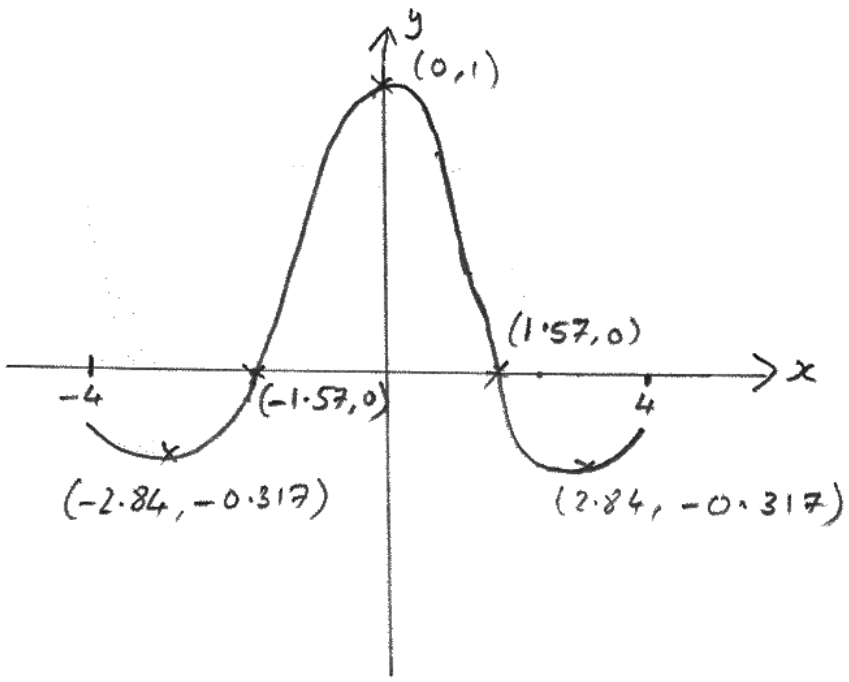
$= 1 - 0.018\ldots$

$= 0.982$ ***A1***

[2 marks]

Total [5 marks]

6. (a)



A1A1A1A1

Note: Award ***A1*** for correct shape. Do not penalise if too large a domain is used,
A1 for correct *x*-intercepts,
A1 for correct coordinates of two minimum points,
A1 for correct coordinates of maximum point.

Accept answers which correctly indicate the position of the intercepts, maximum point and minimum points.

[4 marks]

(b) gradient at $x = 1$ is -0.786 ***A1***

[1 mark]

(c) gradient of normal is $\frac{-1}{-0.786}(=1.272\ldots)$ ***(A1)***

when $x = 1$, $y = 0.3820...$ ***(A1)***

Equation of normal is $y - 0.382 = 1.27(x-1)$ ***A1***

$(\Rightarrow y = 1.27x - 0.890)$

[3 marks]

Total [8 marks]

continued …

7. (a) $\int_0^a \frac{1}{1+x^4}\,dx = 1$ ***M2***

$a = 1.40$ ***A1***

[3 marks]

(b) $E(X) = \int_0^a \frac{x}{1+x^4}\,dx$ ***M1***

$\left(= \frac{1}{2}\arctan(a^2)\right)$

$= 0.548$ ***A1***

[2 marks]

Total [5 marks]

8. (a) height $= 4 \times 0.95^4$ ***(A1)***

$= 3.26$ (metres) ***A1***

[2 marks]

(b) $4 \times 0.95^n < 1$ ***(M1)***

$0.95^n < 0.25$

$\Rightarrow n > \dfrac{\ln 0.25}{\ln 0.95}$ ***(A1)***

$\Rightarrow n > 27.0$

Note: Do not penalize improper use of inequalities.

$\Rightarrow n = 28$ ***A1***

Note: If candidates have used $n-1$ rather than n throughout penalise in part (a) and treat as follow through in parts (b) and (c).

[3 marks]

(c) **METHOD 1**

recognition of geometric series with sum to infinity, first term of 4×0.95 and common ratio 0.95 ***M1***

recognition of the need to double this series and to add 4 ***M1***

total distance travelled is $2\left(\dfrac{4 \times 0.95}{1-0.95}\right) + 4 = 156$ (metres) ***A1***

[3 marks]

Note: If candidates have used $n-1$ rather than n throughout penalise in part (a) and treat as follow through in parts (b) and (c).

METHOD 2

recognition of a geometric series with sum to infinity, first term of 4 and common ratio 0.95 **M1**

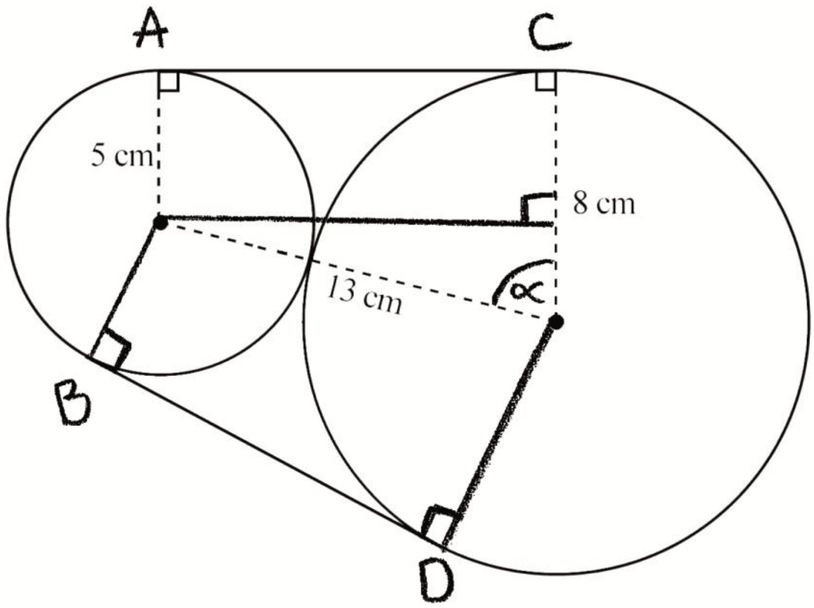
recognition of the need to double this series and to subtract 4 **M1**

total distance travelled is $2\left(\dfrac{4}{1-0.95}\right) - 4 = 156$ (metres) **A1**

[3 marks]

Total [8 marks]

9.



$AC = BD = \sqrt{13^2 - 3^2} = 12.64\ldots$ ***(A1)***

$\cos\alpha = \frac{3}{13} \Rightarrow \alpha = 1.337...\left(76.65...^\circ.\right)$ ***(M1)(A1)***

attempt to find either arc length AB or arc length CD ***(M1)***

arc length $AB = 5(\pi - 2\times 0.232\ldots)\ (= 13.37\ldots)$ ***(A1)***

arc length $CD = 8(\pi + 2\times 0.232\ldots)\ (= 28.85\ldots)$ ***(A1)***

length of string $= 13.37\ldots + 28.85\ldots + 2(12.64\ldots)$ ***(M1)***

$= 67.5\,(\text{cm})$ ***A1***

[8 marks]

SECTION B

10. (a) (i) $P(X > 225) = 0.158\ldots$ ***(M1)(A1)***

expected number $= 450 \times 0.158\ldots = 71.4$ ***A1***

(ii) $P(X < m) = 0.7$ ***(M1)***

$\Rightarrow m = 213$ (grams) ***A1***

[5 marks]

(b) $\dfrac{270-\mu}{\sigma} = 1.40\ldots$ ***(M1)A1***

$\dfrac{250-\mu}{\sigma} = -1.03\ldots$ ***A1***

Note: These could be seen in graphical form.

solving simultaneously ***(M1)***

$\mu = 258,\ \sigma = 8.19$ ***A1A1***

[6 marks]

(c)

$X \sim N(80, 4^2)$

$P(X > 82) = 0.3085\ldots$ ***A1***

recognition of the use of binomial distribution. ***(M1)***

$X \sim B(5, 0.3085\ldots)$

$P(X = 3) = 0.140$ ***A1***

[3 marks]

.

Total [14 marks]

11. (a) in augmented matrix form $\left|\begin{array}{cccc}1 & -3 & 1 & 3\\ 1 & 5 & -2 & 1\\ 0 & 16 & -6 & k\end{array}\right|$

attempt to find a line of zeros ***(M1)***

$r_2 - r_1 \left|\begin{array}{cccc}1 & -3 & 1 & 3\\ 0 & 8 & -3 & -2\\ 0 & 16 & -6 & k\end{array}\right|$ ***(A1)***

$r_3 - 2r_2 \left|\begin{array}{cccc}1 & -3 & 1 & 3\\ 0 & 8 & -3 & -2\\ 0 & 0 & 0 & k+4\end{array}\right|$ ***(A1)***

there is an infinite number of solutions when $k = -4$ ***R1***

there is no solution when

$k \neq -4, (k \in \mathbb{R})$ ***R1***

Note:	Approaches other than using the augmented matrix are acceptable.

[5 marks]

(b) using $\left|\begin{array}{cccc}1 & -3 & 1 & 3\\ 0 & 8 & -3 & -2\\ 0 & 0 & 0 & k+4\end{array}\right|$ and letting $z = \lambda$ ***(M1)***

$8y - 3\lambda = -2$

$\Rightarrow y = \dfrac{3\lambda - 2}{8}$ ***(A1)***

$x - 3y + z = 3$

$\Rightarrow x - \left(\dfrac{9\lambda - 6}{8}\right) + \lambda = 3$ ***(M1)***

$\Rightarrow 8x - 9\lambda + 6 + 8\lambda = 24$

$\Rightarrow x = \dfrac{18 + \lambda}{8}$ ***(A1)***

$\Rightarrow \begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}\frac{18}{8}\\ -\frac{2}{8}\\ 0\end{pmatrix} + \lambda \begin{pmatrix}\frac{1}{8}\\ \frac{3}{8}\\ 1\end{pmatrix}$ ***(M1)(A1)***

$\boldsymbol{r} = \begin{pmatrix}\frac{9}{4}\\ -\frac{1}{4}\\ 0\end{pmatrix} + \lambda \begin{pmatrix}1\\ 3\\ 8\end{pmatrix}$ ***A1***

Note:	Accept equivalent answers.

[7 marks]

continued...

Question 11 continued

(c) recognition that $\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ is parallel to the plane ***(A1)***

direction normal of the plane is given by $\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 3 & 8 \\ 3 & -2 & 0 \end{vmatrix}$ ***(M1)***

$= 16\boldsymbol{i} + 24\boldsymbol{j} - 11\boldsymbol{k}$ ***A1***

Cartesian equation of the plane is given by $16x + 24y - 11z = d$ and a point which fits this equation is $(1, 2, 0)$ ***(M1)***

$\Rightarrow 16 + 48 = d$

$d = 64$ ***A1***

hence Cartesian equation of plane is $16x + 24y - 11z = 64$ ***AG***

Note:	Accept alternative methods using dot product.

[5 marks]

(d) the plane crosses the *z*-axis when $x = y = 0$ ***(M1)***

coordinates of P are $\left(0, 0, -\frac{64}{11}\right)$ ***A1***

Note:	Award ***A1*** for stating $z = -\frac{64}{11}$.

Note:	Accept. $\begin{pmatrix} 0 \\ 0 \\ -\frac{64}{11} \end{pmatrix}$

[2 marks]

(e) recognition that the angle between the line and the direction normal is given by:

$\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 16 \\ 24 \\ -11 \end{pmatrix} = \sqrt{29}\sqrt{953}\cos\theta$ where θ is the angle between the line and the normal vector ***M1A1***

$\Rightarrow 122 = \sqrt{29}\sqrt{953}\cos\theta$ ***(A1)***

$\Rightarrow \theta = 42.8^\circ$ (0.747 radians) ***(A1)***

hence the angle between the line and the plane is $90^\circ - 42.8^\circ = 47.2^\circ$ (0.824 radians) ***A1***

[5 marks]

Note:	Accept use of the formula $\boldsymbol{a}.\boldsymbol{b} = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\sin\theta$.

Total [24 marks]

12. (a) $\dfrac{\mathrm{d}v}{\mathrm{d}t} = -v^2 - 1$

attempt to separate the variables ***M1***

$\int \dfrac{1}{1+v^2}\,\mathrm{d}v = \int -1\,\mathrm{d}t$ ***A1***

$\arctan v = -t + k$ ***A1A1***

Note:	Do not penalize the lack of constant at this stage.

when $t = 0,\ v = 1$ ***M1***

$\Rightarrow k = \arctan 1 = \left(\dfrac{\pi}{4}\right) = \left(45^\circ\right)$ ***A1***

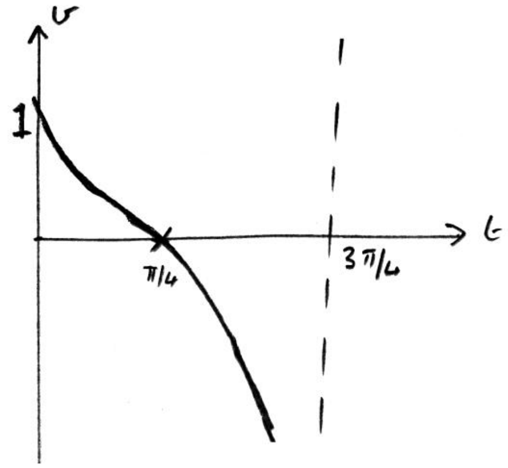
$\Rightarrow v = \tan\left(\dfrac{\pi}{4} - t\right)$ ***A1***

[7 marks]

continued...

Question 12 continued

(b)



A1A1A1

Note:	Award ***A1*** for general shape, ***A1*** for asymptote, ***A1*** for correct *t* and *v* intercept.

Note:	Do not penalise if a larger domain is used.

[3 marks]

(c) (i) $T = \frac{\pi}{4}$ ***A1***

(ii) area under curve $= \int_0^{\frac{\pi}{4}} \tan\left(\frac{\pi}{4} - t\right) dt$ ***(M1)***

$= 0.347 \left(= \frac{1}{2}\ln 2\right)$ ***A1***

[3 marks]

continued...

Question 12 continued

(d) $v = \tan\left(\frac{\pi}{4} - t\right)$

$s = \int \tan\left(\frac{\pi}{4} - t\right) dt$ ***M1***

$\int \frac{\sin\left(\frac{\pi}{4} - t\right)}{\cos\left(\frac{\pi}{4} - t\right)} dt$ ***(M1)***

$= \ln\cos\left(\frac{\pi}{4} - t\right) + k$ ***A1***

when $t = 0$, $s = 0$

$k = -\ln\cos\frac{\pi}{4}$ ***A1***

$s = \ln\cos\left(\frac{\pi}{4} - t\right) - \ln\cos\frac{\pi}{4} \quad \left(= \ln\left[\sqrt{2}\cos\left(\frac{\pi}{4} - t\right)\right]\right)$ ***A1***

[5 marks]

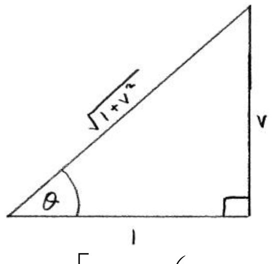
(e) **METHOD 1**

$\frac{\pi}{4} - t = \arctan v$ ***M1***

$t = \frac{\pi}{4} - \arctan v$

$s = \ln\left[\sqrt{2}\cos\left(\frac{\pi}{4} - \frac{\pi}{4} + \arctan v\right)\right]$

$s = \ln\left[\sqrt{2}\cos(\arctan v)\right]$ ***M1A1***



$s = \ln\left[\sqrt{2}\cos\left(\arccos\frac{1}{\sqrt{1+v^2}}\right)\right]$ ***A1***

$= \ln\frac{\sqrt{2}}{\sqrt{1+v^2}}$

$= \frac{1}{2}\ln\frac{2}{1+v^2}$ ***AG***

continued...

Question 12 continued

METHOD 2

$s = \ln\cos\left(\frac{\pi}{4} - t\right) - \ln\cos\frac{\pi}{4}$

$= -\ln\sec\left(\frac{\pi}{4} - t\right) - \ln\cos\frac{\pi}{4}$ ***M1***

$= -\ln\sqrt{1 + \tan^2\left(\frac{\pi}{4} - t\right)} - \ln\cos\frac{\pi}{4}$ ***M1***

$= -\ln\sqrt{1 + v^2} - \ln\cos\frac{\pi}{4}$ ***A1***

$= \ln\frac{1}{\sqrt{1 + v^2}} + \ln\sqrt{2}$ ***A1***

$= \frac{1}{2}\ln\frac{2}{1 + v^2}$ ***AG***

METHOD 3

$v\frac{dv}{ds} = -v^2 - 1$ ***M1***

$\int\frac{v}{v^2 + 1}dv = -\int 1ds$ ***M1***

$\frac{1}{2}\ln(v^2 + 1) = -s + k$ ***A1***

when $s = 0$, $t = 0 \Rightarrow v = 1$

$\Rightarrow k = \frac{1}{2}\ln 2$ ***A1***

$\Rightarrow s = \frac{1}{2}\ln\frac{2}{1 + v^2}$ ***AG***

[4 marks]

Total [22 marks]