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MARKSCHEME

May 2011

MATHEMATICS

Higher Level

Paper 2

15 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question. Award the marks as usual and then write –1(****MR****) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled **(d)** *and a brief* ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

Candidates should be penalized ***once only IN THE PAPER*** *for an accuracy error* ***(AP)****. Award the marks as usual then write* ***(AP)*** *against the answer. On the* ***front*** *cover write –1****(AP)****. Deduct 1 mark from the total for the paper, not the question.*

- If a final correct answer is incorrectly rounded, apply the ***AP***.
- If the level of accuracy is not specified in the question, apply the ***AP*** for correct answers not given to three significant figures.

If there is no working shown, and answers are given to the correct two significant figures, apply the ***AP***. However, do **not** accept answers to one significant figure without working.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

SECTION A

1. area of triangle POQ $= \frac{1}{2}8^2 \sin 59°$ *M1*

$= 27.43$ *(A1)*

area of sector $= \pi 8^2 \frac{59}{360}$ *M1*

$= 32.95$ *(A1)*

area between arc and chord $= 32.95 - 27.43$

$= 5.52 \ (\text{cm}^2)$ *A1*

[5 marks]

2. $u_4 = u_1 + 3d = 7$, $u_9 = u_1 + 8d = 22$ *A1A1*

Note: 5d=15 gains both above marks

$u_1 = -2$, $d = 3$ *A1*

$S_n = \frac{n}{2}\left(-4 + (n-1)3\right) > 10\,000$ *M1*

$n = 83$ *A1*

[5 marks]

3. (a) $a = 10\mathrm{e}^{-0.2t}$ *(M1)(A1)*

at $t = 10$, $a = 1.35 \ (\text{m s}^{-2})$ (accept $10e^{-2}$) *A1*

(b) **METHOD 1**

$d = \int_0^{10} 50(1 - \mathrm{e}^{-0.2t})\,\mathrm{d}t$ *(M1)*

$= 283.83...$ *A1*

so distance above ground $= 1720$ (m) (3 sf) (accept 1716 (m)) *A1*

METHOD 2

$\text{s} = \int 50(1 - \mathrm{e}^{-0.2t})\,\mathrm{d}t = 50t + 250e^{-0.2t} (+c)$ *M1*

Taking $s = 0$ when $t = 0$ gives $c = -250$ *M1*

So when $t = 10$, $s = 283.3...$

so distance above ground $= 1720$ (m) (3 sf) (accept 1716 (m)) *A1*

[6 marks]

4. (a) $\det \boldsymbol{A} = \cos 2\theta \cos\theta + \sin 2\theta \sin\theta$ ***M1A1***

$= \cos(2\theta - \theta)$ ***A1***

Note: Allow use of double angle formulae if they lead to the correct answer

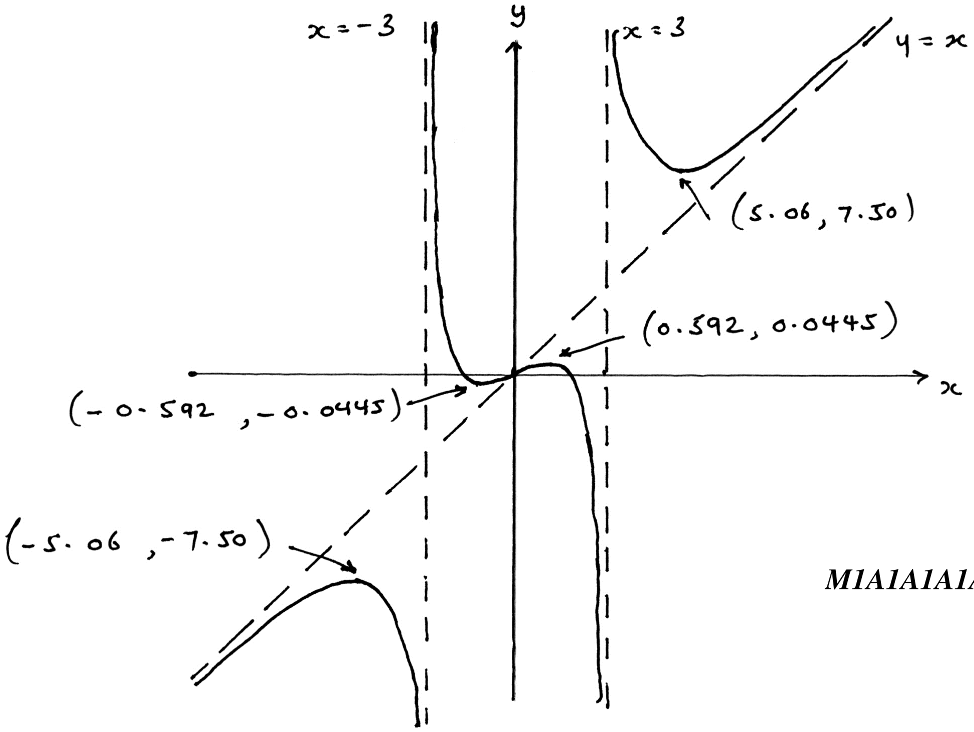
$= \cos\theta$ ***AG***

(b) $\cos^2\theta = \sin\theta$ ***A1***

$\theta = 0.666,\ 2.48$ ***A1A1***

[6 marks]

5.



M1A1A1A1A1A1A1

Note: Award ***A1*** for both vertical asymptotes correct,
M1 for recognizing that there are two turning points near the origin,
A1 for both turning points near the origin correct, (only this ***A*** mark is dependent on the ***M*** mark)
A1 for the other pair of turning points correct,
A1 for correct positioning of the oblique asymptote,
A1 for correct equation of the oblique asymptote,
A1 for correct asymptotic behaviour in all sections.

[7 marks]

6. (a) $\mathrm{P}(x<1.4)=0.691$ (accept 0.692) ***A1***

(b) **METHOD 1**

$Y \sim \mathrm{B}(6,\ 0.3085...)$ ***(M1)***

$\mathrm{P}(Y \geq 4)=1-\mathrm{P}(Y \leq 3)$ ***(M1)***

$=0.0775$ (accept 0.0778 if 3sf approximation from (a) used) ***A1***

METHOD 2

$X \sim \mathrm{B}(6,\ 0.6914...)$ ***(M1)***

$\mathrm{P}(X \leq 2)$ ***(M1)***

$=0.0775$ (accept 0.0778 if 3sf approximation from (a) used) ***A1***

(c) $\mathrm{P}(x<1 \mid x<1.4)=\dfrac{\mathrm{P}(x<1)}{\mathrm{P}(x<1.4)}$ ***M1***

$=\dfrac{0.06680...}{0.6914...}$

$=0.0966$ (accept 0.0967) ***A1***

[6 marks]

7. (a) $x^3+1=\dfrac{1}{x^3+1}$

$(-1.26,\ -1)$ $\left(=\left(-\sqrt[3]{2},\ -1\right)\right)$ ***A1***

(b) $f'(-1.259...)=4.762...$ $(3\times 2^{\frac{2}{3}})$ ***A1***

$g'(-1.259...)=-4.762...$ $(-3\times 2^{\frac{2}{3}})$ ***A1***

required angle $=2\arctan\left(\dfrac{1}{4.762...}\right)$ ***M1***

$=0.414$ (accept $23.7°$) ***A1***

Note: Accept alternative methods including finding the obtuse angle first.

[5 marks]

8. let the length of one side of the triangle be x
consider the triangle consisting of a side of the triangle and two radii

EITHER

$x^2 = r^2 + r^2 - 2r^2 \cos 120°$ ***M1***

$= 3r^2$

OR

$x = 2r \cos 30°$ ***M1***

THEN

$x = r\sqrt{3}$ ***A1***

so perimeter $= 3\sqrt{3}\, r$ ***A1***

now consider the area of the triangle

area $= 3 \times \frac{1}{2} r^2 \sin 120°$ ***M1***

$= 3 \times \frac{\sqrt{3}}{4} r^2$ ***A1***

$\frac{P}{A} = \frac{3\sqrt{3}\, r}{\frac{3\sqrt{3}}{4} r^2}$

$= \frac{4}{r}$ ***A1***

Note: Accept alternative methods

[6 marks]

9. let $x =$ distance from observer to rocket
let $h =$ the height of the rocket above the ground

METHOD 1

$\frac{dh}{dt} = 300$ when $h = 800$ ***A1***

$x = \sqrt{h^2 + 360\,000} = (h^2 + 360\,000)^{\frac{1}{2}}$ ***M1***

$\frac{dx}{dh} = \frac{h}{\sqrt{h^2 + 360\,000}}$ ***A1***

when $h = 800$

$\frac{dx}{dt} = \frac{dx}{dh} \times \frac{dh}{dt}$ ***M1***

$= \frac{300h}{\sqrt{h^2 + 360\,000}}$ ***A1***

$= 240\ (\text{m s}^{-1})$ ***A1***

[6 marks]

METHOD 2

$h^2 + 600^2 = x^2$ ***M1***

$2h = 2x\frac{dx}{dh}$ ***A1***

$\frac{dx}{dh} = \frac{h}{x}$

$= \frac{800}{1000}\ \left(= \frac{4}{5}\right)$ ***A1***

$\frac{dh}{dt} = 300$ ***A1***

$\frac{dx}{dt} = \frac{dx}{dh} \times \frac{dh}{dt}$ ***M1***

$= \frac{4}{5} \times 300$

$= 240\ (\text{m s}^{-1})$ ***A1***

[6 marks]

METHOD 3

$x^2 = 600^2 + h^2$ ***M1***

$2x\frac{dx}{dt} = 2h\frac{dh}{dt}$ ***A1A1***

when $h = 800$, $x = 1000$

$\frac{dx}{dt} = \frac{800}{1000} \times \frac{dh}{dt}$ ***M1A1***

$= 240\ \text{m s}^{-1}$ ***A1***

continued …

Question 9 continued

METHOD 4

Distance between the observer and the rocket = $(600^2 + 800^2)^{\frac{1}{2}} = 1000$ ***M1A1***

Component of the velocity in the line of sight = $\sin\theta \times 300$
(where θ = angle of elevation) ***M1A1***

$\sin\theta = \dfrac{800}{1000}$ ***A1***

component = 240 (ms^{-1}) ***A1***

[6 marks]

10. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$

$\dfrac{1}{2}x^{-\frac{1}{2}} + \dfrac{1}{2}y^{-\frac{1}{2}}\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1***

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}} = -\sqrt{\dfrac{y}{x}}$ ***A1***

Note: Accept $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 - \dfrac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$ from making *y* the subject of the equation, and all correct subsequent working

therefore the gradient at the point P is given by

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{\dfrac{q}{p}}$ ***A1***

equation of tangent is $y - q = -\sqrt{\dfrac{q}{p}}(x - p)$ ***M1***

$\left(y = -\sqrt{\dfrac{q}{p}}x + q + \sqrt{q}\sqrt{p}\right)$

x-intercept: $y = 0$, $n = \dfrac{q\sqrt{p}}{\sqrt{q}} + p = \sqrt{q}\sqrt{p} + p$ ***A1***

y-intercept: $x = 0$, $m = \sqrt{q}\sqrt{p} + q$ ***A1***

$n + m = \sqrt{q}\sqrt{p} + p + \sqrt{q}\sqrt{p} + q$ ***M1***

$= 2\sqrt{q}\sqrt{p} + p + q$

$= \left(\sqrt{p} + \sqrt{q}\right)^2$ ***A1***

$= a$ ***AG***

[8 marks]

SECTION B

11. (a) $\vec{PQ} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}, \vec{SR} = \begin{pmatrix} 0-x \\ 5-y \\ 1-z \end{pmatrix}$ ***(M1)***

point S $= (1, 6, -2)$ ***A1***

[2 marks]

(b) $\vec{PQ} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} \quad \vec{PS} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ ***A1***

$\vec{PQ} \times \vec{PS} = \begin{pmatrix} -13 \\ 7 \\ -2 \end{pmatrix}$

$m = -2$ ***A1***

[2 marks]

(c) area of parallelogram PQRS $= \left| \vec{PQ} \times \vec{PS} \right| = \sqrt{(-13)^2 + 7^2 + (-2)^2}$ ***M1***

$= \sqrt{222} = 14.9$ ***A1***

[2 marks]

(d) equation of plane is $-13x + 7y - 2z = d$ ***M1A1***

substituting any of the points given gives $d = 33$

$-13x + 7y - 2z = 33$ ***A1***

[3 marks]

(e) equation of line is $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -13 \\ 7 \\ -2 \end{pmatrix}$ ***A1***

Note: To get the ***A1*** must have $\boldsymbol{r} =$ or equivalent.

[1 mark]

(f) $169\lambda + 49\lambda + 4\lambda = 33$ ***M1***

$\lambda = \frac{33}{222} \quad (= 0.149...)$ ***A1***

closest point is $\left(-\frac{143}{74}, \frac{77}{74}, -\frac{11}{37} \right) \quad (= (-1.93, 1.04, -0.297))$ ***A1***

[3 marks]

(g) angle between planes is the same as the angle between the normals ***(R1)***

$\cos\theta = \frac{-13 \times 1 + 7 \times -2 - 2 \times 1}{\sqrt{222} \times \sqrt{6}}$ ***M1A1***

$\theta = 143^\circ$ (accept $\theta = 37.4^\circ$ or 2.49 radians or 0.652 radians) ***A1***

[4 marks]

Total [17 marks]

12. (a) $\mathrm{P}(x=0)=0.607$ ***A1***

[1 mark]

(b) **EITHER**

Using $X \sim \mathrm{Po}(3)$ ***(M1)***

OR

Using $(0.6065...)^6$ ***(M1)***

THEN

$\mathrm{P}(X=0)=0.0498$ ***A1***

[2 marks]

(c) $X \sim \mathrm{Po}(0.5t)$ ***(M1)***

$\mathrm{P}(x \geq 1)=1-\mathrm{P}(x=0)$ ***(M1)***

$\mathrm{P}(x=0)<0.01$ ***A1***

$\mathrm{e}^{-0.5t}<0.01$ ***A1***

$-0.5t<\ln(0.01)$ ***(M1)***

$t>9.21$ months

therefore 10 months ***A1N4***

Note:	Full marks can be awarded for answers obtained directly from GDC if a systematic method is used and clearly shown.

[6 marks]

(d) (i) P(1 or 2 accidents) = 0.37908… ***A1***

$\mathrm{E}(B)=1000\times 0.60653...+500\times 0.37908...$ ***M1A1***

$=\$796$ (accept \$797 or \$796.07) ***A1***

(ii) $\mathrm{P}(2000)=\mathrm{P}(1000, 1000, 0)+\mathrm{P}(1000, 0, 1000)+\mathrm{P}(0, 1000, 1000)+$
$\mathrm{P}(1000, 500, 500)+\mathrm{P}(500, 1000, 500)+\mathrm{P}(500, 500, 1000)$ ***(M1)(A1)***

Note:	Award ***M1*** for noting that 2000 can be written both as $2\times 1000+1\times 0$ and $2\times 500+1\times 1000$.

$=3(0.6065...)^2(0.01437...)+3(0.3790...)^2(0.6065...)$ ***M1A1***

$=0.277$ (accept 0.278) ***A1***

[9 marks]

Total [18 marks]

13. Part A

prove that $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+...+n\left(\frac{1}{2}\right)^{n-1}=4-\frac{n+2}{2^{n-1}}$

for $n=1$

$\text{LHS}=1,\ \text{RHS}=4-\frac{1+2}{2^0}=4-3=1$

so true for $n=1$ ***R1***

assume true for $n=k$ ***M1***

so $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+...+k\left(\frac{1}{2}\right)^{k-1}=4-\frac{k+2}{2^{k-1}}$

now for $n=k+1$

LHS: $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+...+k\left(\frac{1}{2}\right)^{k-1}+(k+1)\left(\frac{1}{2}\right)^k$ ***A1***

$=4-\frac{k+2}{2^{k-1}}+(k+1)\left(\frac{1}{2}\right)^k$ ***M1A1***

$=4-\frac{2(k+2)}{2^k}+\frac{k+1}{2^k}$ (or equivalent) ***A1***

$=4-\frac{(k+1)+2}{2^{(k+1)-1}}$ (accept $4-\frac{k+3}{2^k}$) ***A1***

Therefore if it is true for $n=k$ it is true for $n=k+1$. It has been shown to be true for $n=1$ so it is true for all n ($\in \mathbb{Z}^+$). ***R1***

Note: To obtain the final ***R*** mark, a reasonable attempt at induction must have been made.

[8 marks]

Part B

(a) **METHOD 1**

$\int e^{2x}\sin x\,dx=-\cos x e^{2x}+\int 2e^{2x}\cos x\,dx$ ***M1A1A1***

$=-\cos x e^{2x}+2e^{2x}\sin x-\int 4e^{2x}\sin x\,dx$ ***A1A1***

$5\int e^{2x}\sin x\,dx=-\cos x e^{2x}+2e^{2x}\sin x$ ***M1***

$\int e^{2x}\sin x\,dx=\frac{1}{5}e^{2x}(2\sin x-\cos x)+C$ ***AG***

METHOD 2

$\int \sin x e^{2x}\,dx=\frac{\sin x e^{2x}}{2}-\int \cos x\frac{e^{2x}}{2}dx$ ***M1A1A1***

$=\frac{\sin x e^{2x}}{2}-\cos x\frac{e^{2x}}{4}-\int \sin x\frac{e^{2x}}{4}dx$ ***A1A1***

$\frac{5}{4}\int e^{2x}\sin x\,dx=\frac{e^{2x}\sin x}{2}-\frac{\cos x e^{2x}}{4}$ ***M1***

$\int e^{2x}\sin x\,dx=\frac{1}{5}e^{2x}(2\sin x-\cos x)+C$ ***AG***

[6 marks]

continued …

Question 13 continued

(b) $\int \frac{dy}{\sqrt{1-y^2}} = \int e^{2x} \sin x \, dx$ ***M1A1***

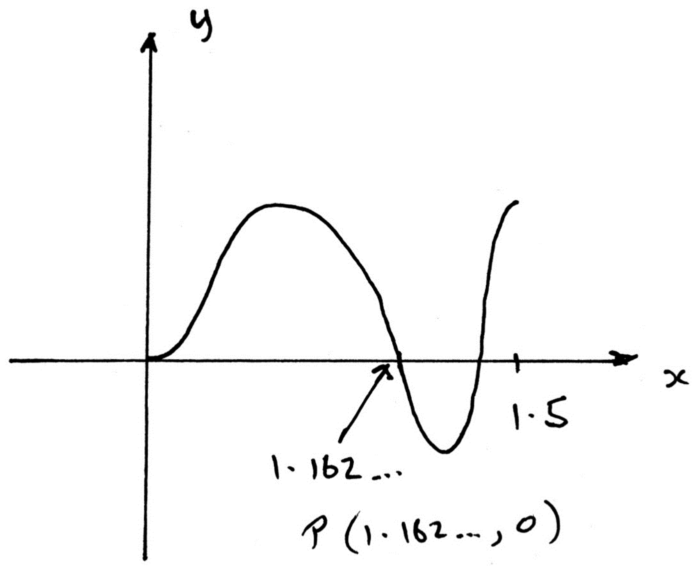
$\arcsin y = \frac{1}{5} e^{2x} (2\sin x - \cos x) (+C)$ ***A1***

when $x = 0,\ y = 0 \Rightarrow C = \frac{1}{5}$ ***M1***

$y = \sin\left(\frac{1}{5} e^{2x} (2\sin x - \cos x) + \frac{1}{5}\right)$ ***A1***

[5 marks]

(c) (i)



A1

P is $(1.16, 0)$ ***A1***

Note:	Award ***A1*** for 1.16 seen anywhere, ***A1*** for complete sketch.

Note:	Allow FT on their answer from (b)

(ii) $V = \int_0^{1.162...} \pi y^2 \, dx$ ***M1A1***

$= 1.05$ ***A2***

Note:	Allow FT on their answers from (b) and (c)(i).

[6 marks]

Total [25 marks]