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MATHEMATICS
HIGHER LEVEL
PAPER 3 – SERIES AND DIFFERENTIAL EQUATIONS

Thursday 20 May 2010 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 8]*

Given that $\frac{\mathrm{d}y}{\mathrm{d}x} - 2y^2 = \mathrm{e}^x$ and $y = 1$ when $x = 0$, use Euler's method with a step length of 0.1 to find an approximation for the value of y when $x = 0.4$. Give all intermediate values with maximum possible accuracy.

2. *[Maximum mark: 11]*

(a) Using integration by parts, show that $\int_0^\infty \mathrm{e}^{-x} \cos x \,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-x} \sin x \,\mathrm{d}x$. *[5 marks]*

(b) Find the value of these two integrals. *[6 marks]*

3. *[Maximum mark: 9]*

Solve the differential equation

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + xy + 4x^2,$$

given that $y = 2$ when $x = 1$. Give your answer in the form $y = f(x)$.

4. *[Maximum mark: 17]*

(a) Using the Maclaurin series for $(1+x)^n$, write down and simplify the Maclaurin series approximation for $(1-x^2)^{-\frac{1}{2}}$ as far as the term in x^4. *[3 marks]*

(b) Use your result to show that a series approximation for $\arccos x$ is

$$\arccos x \approx \frac{\pi}{2} - x - \frac{1}{6}x^3 - \frac{3}{40}x^5 .$$ *[3 marks]*

(c) Evaluate $\lim_{x \to 0} \frac{\frac{\pi}{2} - \arccos(x^2) - x^2}{x^6}$. *[5 marks]*

(d) Use the series approximation for $\arccos x$ to find an approximate value for

$$\int_0^{0.2} \arccos\left(\sqrt{x}\right) \mathrm{d}x ,$$

giving your answer to 5 decimal places. Does your answer give the actual value of the integral to 5 decimal places? *[6 marks]*

5. *[Maximum mark: 15]*

(a) Consider the power series $\sum_{k=1}^{\infty} k\left(\frac{x}{2}\right)^k$.

(i) Find the radius of convergence.

(ii) Find the interval of convergence. *[10 marks]*

(b) Consider the infinite series $\sum_{k=1}^{\infty} (-1)^{k+1} \times \frac{k}{2k^2+1}$.

(i) Show that the series is convergent.

(ii) Show that the sum to infinity of the series is less than 0.25. *[5 marks]*