22107206

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 2

Thursday 6 May 2010 (morning)

2 hours

Candidate session number

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INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** the questions in the spaces provided. Working may be continued below the lines, if necessary.*

1. *[Maximum mark: 4]*

Consider the arithmetic sequence $8, 26, 44, \ldots$.

(a) Find an expression for the n^{th} term. *[1 mark]*

(b) Write down the sum of the first n terms using sigma notation. *[1 mark]*

(c) Calculate the sum of the first 15 terms. *[2 marks]*

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2. *[Maximum mark: 6]*

A discrete random variable X has a probability distribution given in the following table.

x	0.5	1.5	2.5	3.5	4.5	5.5
$\mathrm{P}(X = x)$	0.15	0.21	p	q	0.13	0.07

(a) If $\mathrm{E}(X) = 2.61$, determine the value of p and of q. *[4 marks]*

(b) Calculate $\mathrm{Var}(X)$ to three significant figures. *[2 marks]*

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3. *[Maximum mark: 5]*

After being sprayed with a weedkiller, the survival time of weeds in a field is normally distributed with a mean of 15 days.

(a) If the probability of survival after 21 days is 0.2, find the standard deviation of the survival time. *[3 marks]*

When another field is sprayed, the survival time of weeds is normally distributed with a mean of 18 days.

(b) If the standard deviation of the survival time is unchanged, find the probability of survival after 21 days. *[2 marks]*

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4. *[Maximum mark: 6]*

(a) Find the solution of the equation

$$\ln 2^{4x-1} = \ln 8^{x+5} + \log_2 16^{1-2x},$$

expressing your answer in terms of $\ln 2$. *[4 marks]*

(b) Using this value of x, find the value of a for which $\log_a x = 2$, giving your answer to three decimal places. *[2 marks]*

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5. *[Maximum mark: 6]*

Consider the triangle ABC where $B\hat{A}C = 70^\circ$, $AB = 8\,\text{cm}$ and $AC = 7\,\text{cm}$. The point D on the side BC is such that $\frac{BD}{DC} = 2$.

Determine the length of AD.

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6. *[Maximum mark: 6]*

The random variable X follows a Poisson distribution with mean λ.

(a) Find λ if $\mathrm{P}(X=0)+\mathrm{P}(X=1)=0.123$. *[4 marks]*

(b) With this value of λ, find $\mathrm{P}(0<X<9)$. *[2 marks]*

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7. *[Maximum mark: 6]*

The three planes

$$2x - 2y - z = 3$$
$$4x + 5y - 2z = -3$$
$$3x + 4y - 3z = -7$$

intersect at the point with coordinates (a, b, c).

(a) Find the value of each of a, b and c. *[2 marks]*

(b) The equations of three other planes are

$$2x - 4y - 3z = 4$$
$$-x + 3y + 5z = -2$$
$$3x - 5y - z = 6.$$

Find a vector equation of the line of intersection of these three planes. *[4 marks]*

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8. *[Maximum mark: 6]*

(a) Simplify the difference of binomial coefficients

$$\binom{n}{3}-\binom{2n}{2}, \text{ where } n \geq 3 .$$ *[4 marks]*

(b) Hence, solve the inequality

$$\binom{n}{3}-\binom{2n}{2} > 32n, \text{ where } n \geq 3 .$$ *[2 marks]*

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9. *[Maximum mark: 7]*

Given that $z = \cos\theta + \mathrm{i}\sin\theta$ show that

(a) $\mathrm{Im}\left(z^n + \frac{1}{z^n}\right) = 0,\ n \in \mathbb{Z}^+$; *[2 marks]*

(b) $\mathrm{Re}\left(\frac{z-1}{z+1}\right) = 0,\ z \neq -1$. *[5 marks]*

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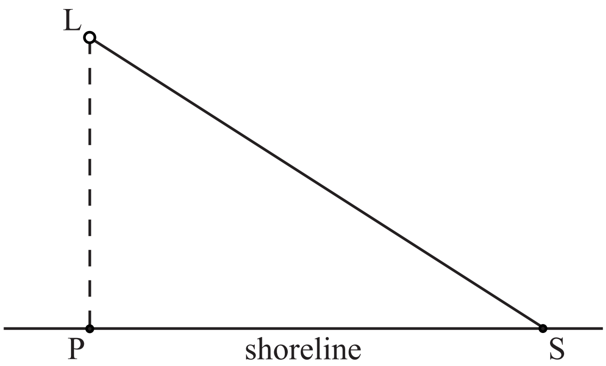
10. *[Maximum mark: 8]*

A lighthouse L is located offshore, 500 metres from the nearest point P on a long straight shoreline. The narrow beam of light from the lighthouse rotates at a constant rate of 8π radians per minute, producing an illuminated spot S that moves along the shoreline. You may assume that the height of the lighthouse can be ignored and that the beam of light lies in the horizontal plane defined by sea level.



When S is 2000 metres from P,

(a) show that the speed of S, correct to three significant figures, is 214 000 metres per minute; *[5 marks]*

(b) find the acceleration of S. *[3 marks]*

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SECTION B

*Answer **all** the questions on the answer sheets provided. Please start each question on a new page.*

11. *[Maximum mark: 14]*

The function f is defined by

$$f(x) = (x^3 + 6x^2 + 3x - 10)^{\frac{1}{2}}, \text{ for } x \in D,$$

where $D \subseteq \mathbb{R}$ is the greatest possible domain of f.

(a) Find the roots of $f(x) = 0$. *[2 marks]*

(b) Hence specify the set D. *[2 marks]*

(c) Find the coordinates of the local maximum on the graph $y = f(x)$. *[2 marks]*

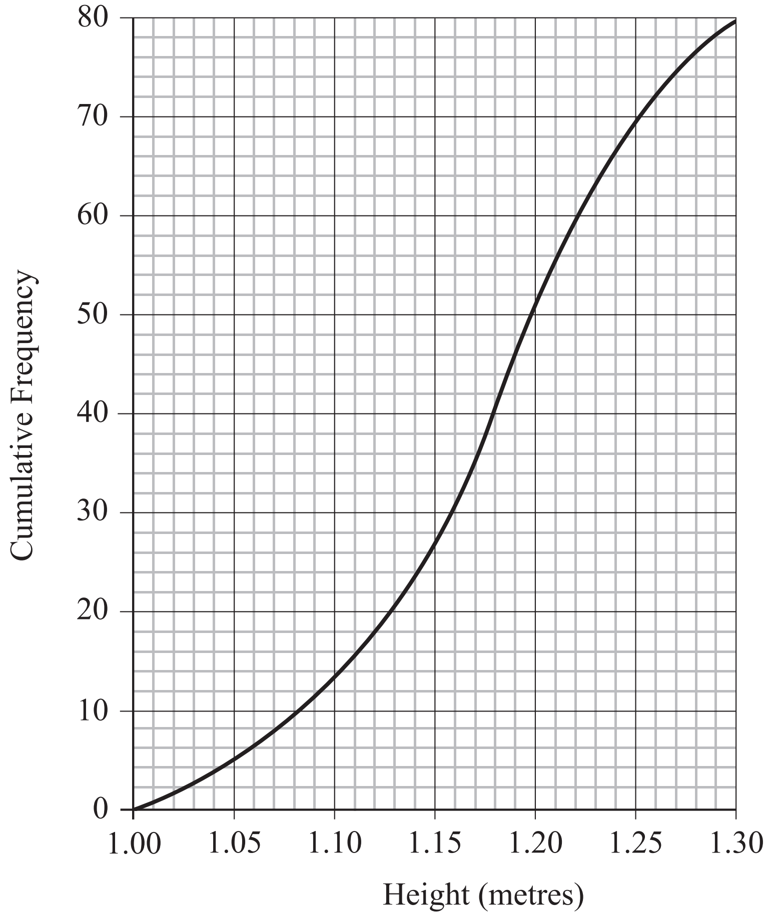
(d) Solve the equation $f(x) = 3$. *[2 marks]*

(e) Sketch the graph of $|y| = f(x)$, for $x \in D$. *[3 marks]*

(f) Find the area of the region completely enclosed by the graph of $|y| = f(x)$. *[3 marks]*

12. *[Maximum mark: 13]*

The heights in metres of a random sample of 80 boys in a certain age group were measured and the following cumulative frequency graph obtained.



(a) (i) Estimate the median of these data.

(ii) Estimate the interquartile range for these data. *[3 marks]*

(b) (i) Produce a frequency table for these data, using a class width of 0.05 metres.

(ii) Calculate unbiased estimates of the mean and variance of the heights of the population of boys in this age group. *[5 marks]*

(c) A boy is selected at random from these 80 boys.

(i) Find the probability that his height is less than or equal to 1.15 metres.

(ii) Given that his height is less than or equal to 1.15 metres, find the probability that his height is less than or equal to 1.12 metres. *[5 marks]*

13. *[Maximum mark: 12]*

The interior of a circle of radius 2 cm is divided into an infinite number of sectors. The areas of these sectors form a geometric sequence with common ratio k. The angle of the first sector is θ radians.

(a) Show that $\theta = 2\pi(1-k)$. *[5 marks]*

(b) The perimeter of the third sector is half the perimeter of the first sector.

Find the value of k and of θ. *[7 marks]*

14. *[Maximum mark: 21]*

The functions f, g and h are defined by

$$f(x) = 1 + \mathrm{e}^x \text{, for } x \in \mathbb{R},$$
$$g(x) = \frac{1}{x} \text{, for } x \in \mathbb{R} / \{0\},$$
$$h(x) = \sec x \text{, for } x \in \mathbb{R} / \left\{\frac{2n+1}{2}\pi,\ n \in \mathbb{Z}\right\}.$$

(a) Determine the range of the composite function $g \circ f$. *[3 marks]*

(b) Determine the inverse of the function $g \circ f$, clearly stating the domain. *[4 marks]*

(c) (i) Show that the function $y = (f \circ g \circ h)(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (1-y)\sin x.$$

(ii) Hence, or otherwise, find $\int y \sin x \,\mathrm{d}x$, as a function of x.

(iii) You are given that the domain of $y = (f \circ g \circ h)(x)$ can be extended to the whole real axis. That part of the graph of $y = (f \circ g \circ h)(x)$, between its maximum at $x = 0$ and its first minimum for positive x, is rotated by 2π about the y-axis. Calculate the volume of the solid generated. *[14 marks]*