N09/5/MATHL/HP1/ENG/TZ0/XX

88097201

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 1

Thursday 5 November 2009 (afternoon)

2 hours

Candidate session number

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INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** the questions in the spaces provided. Working may be continued below the lines, if necessary.*

1. *[Maximum mark: 5]*

When $3x^5 - ax + b$ is divided by $x - 1$ and $x + 1$ the remainders are equal. Given that $a, b \in \mathbb{R}$, find

(a) the value of a; *[4 marks]*

(b) the set of values of b. *[1 mark]*

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2. *[Maximum mark: 5]*

Find the values of n such that $\left(1+\sqrt{3}\,\mathrm{i}\right)^n$ is a real number.

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3. *[Maximum mark: 5]*

Matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are defined as

$$\boldsymbol{A}=\begin{pmatrix}1 & 5 & 1\\ 3 & -1 & 3\\ -9 & 3 & 7\end{pmatrix},\ \boldsymbol{B}=\begin{pmatrix}1 & 2 & -1\\ 3 & -1 & 0\\ 0 & 3 & 1\end{pmatrix},\ \boldsymbol{C}=\begin{pmatrix}8\\ 0\\ -4\end{pmatrix}.$$

(a) Given that $\boldsymbol{AB}=\begin{pmatrix}a & 0 & 0\\ 0 & a & 0\\ 0 & 0 & a\end{pmatrix}$, find a. *[1 mark]*

(b) Hence, or otherwise, find $\boldsymbol{A}^{-1}$. *[2 marks]*

(c) Find the matrix $\boldsymbol{X}$, such that $\boldsymbol{AX}=\boldsymbol{C}$. *[2 marks]*

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4. *[Maximum mark: 6]*

Consider the function f, where $f(x) = \arcsin(\ln x)$.

(a) Find the domain of f. *[3 marks]*

(b) Find $f^{-1}(x)$. *[3 marks]*

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5. *[Maximum mark: 5]*

The real root of the equation $x^3 - x + 4 = 0$ is -1.796 to three decimal places. Determine the real root for each of the following.

(a) $(x-1)^3 - (x-1) + 4 = 0$ *[2 marks]*

(b) $8x^3 - 2x + 4 = 0$ *[3 marks]*

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6. *[Maximum mark: 5]*

At a nursing college, 80 % of incoming students are female. College records show that 70 % of the incoming females graduate and 90 % of the incoming males graduate. A student who graduates is selected at random. Find the probability that the student is male, giving your answer as a fraction in its lowest terms.

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7. *[Maximum mark: 9]*

(a) Calculate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\sqrt[3]{\tan x}}\,\mathrm{d}x$. *[6 marks]*

(b) Find $\int \tan^3 x\,\mathrm{d}x$. *[3 marks]*

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8. *[Maximum mark: 7]*

A certain population can be modelled by the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = k y \cos kt$, where y is the population at time t hours and k is a positive constant.

(a) Given that $y = y_0$ when $t = 0$, express y in terms of k, t and y_0. *[5 marks]*

(b) Find the ratio of the minimum size of the population to the maximum size of the population. *[2 marks]*

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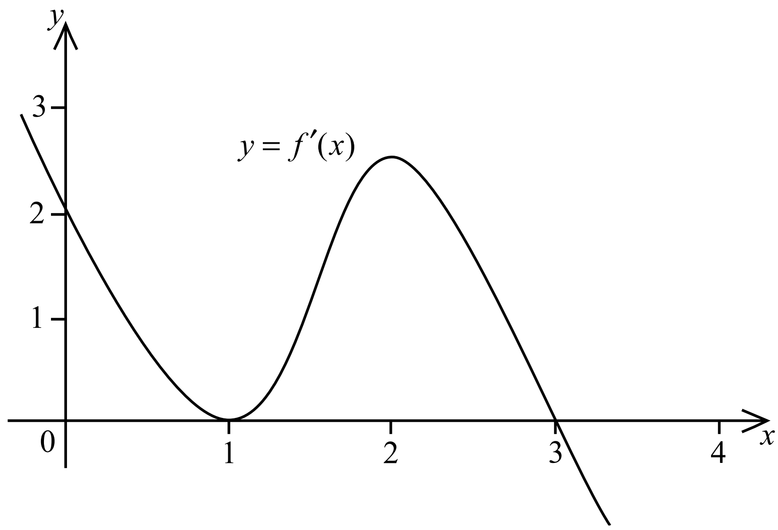
9. *[Maximum mark: 5]*

The diagram below shows a sketch of the gradient function $f'(x)$ of the curve $f(x)$.



On the graph below, sketch the curve $y = f(x)$ given that $f(0) = 0$. Clearly indicate on the graph any maximum, minimum or inflexion points.

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10. *[Maximum mark: 8]*

A drinking glass is modelled by rotating the graph of $y = \mathrm{e}^x$ about the y-axis, for $1 \le y \le 5$. Find the volume of the glass.

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SECTION B

*Answer **all** the questions on the answer sheets provided. Please start each question on a new page.*

11. *[Maximum mark: 21]*

(a) The sum of the first six terms of an arithmetic series is 81. The sum of its first eleven terms is 231. Find the first term and the common difference. *[6 marks]*

(b) The sum of the first two terms of a geometric series is 1 and the sum of its first four terms is 5. If all of its terms are positive, find the first term and the common ratio. *[5 marks]*

(c) The r^{th} term of a new series is defined as the product of the r^{th} term of the arithmetic series and the r^{th} term of the geometric series above. Show that the r^{th} term of this new series is $(r+1)2^{r-1}$. *[3 marks]*

(d) Using mathematical induction, prove that

$$\sum_{r=1}^{n}(r+1)2^{r-1} = n2^n, \ n \in \mathbb{Z}^+.$$ *[7 marks]*

12. *[Maximum mark: 17]*

A tangent to the graph of $y = \ln x$ passes through the origin.

(a) Sketch the graphs of $y = \ln x$ and the tangent on the same set of axes, and hence find the equation of the tangent. *[11 marks]*

(b) Use your sketch to explain why $\ln x \le \frac{x}{\text{e}}$ for $x > 0$. *[1 mark]*

(c) Show that $x^{\text{e}} \le \text{e}^x$ for $x > 0$. *[3 marks]*

(d) Determine which is larger, π^{e} or e^{π}. *[2 marks]*

13. *[Maximum mark: 22]*

(a) Let $z = x + \mathrm{i}y$ be any non-zero complex number.

(i) Express $\frac{1}{z}$ in the form $u + \mathrm{i}v$.

(ii) If $z + \frac{1}{z} = k$, $k \in \mathbb{R}$, show that either $y = 0$ or $x^2 + y^2 = 1$.

(iii) Show that if $x^2 + y^2 = 1$ then $|k| \le 2$. *[8 marks]*

(b) Let $w = \cos\theta + \mathrm{i}\sin\theta$.

(i) Show that $w^n + w^{-n} = 2\cos n\theta$, $n \in \mathbb{Z}$.

(ii) Solve the equation $3w^2 - w + 2 - w^{-1} + 3w^{-2} = 0$, giving the roots in the form $x + \mathrm{i}y$. *[14 marks]*