International Baccalaureate®
Baccalauréat International
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MARKSCHEME

May 2009

MATHEMATICS

Higher Level

Paper 1

Samples to Team Leaders	8 June 2009
Everything (marks, scripts *etc.*) to IB Cardiff	16 June 2009

24 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*., do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question. Award the marks as usual and then write –1(****MR****) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled ***(d)*** *and a brief* ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

*Candidates should be penalized **once only IN THE PAPER** for an accuracy error **(AP)**. Award the marks as usual then write **(AP)** against the answer. On the **front** cover write –1**(AP)**. Deduct 1 mark from the total for the paper, not the question*.

- If a final correct answer is incorrectly rounded, apply the ***AP***.
- If the level of accuracy is not specified in the question, apply the ***AP*** for correct answers not given to three significant figures.

If there is no working shown, and answers are given to the correct two significant figures, apply the ***AP***. However, do **not** accept answers to one significant figure without working.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

SECTION A

1. (a) $|z| = \sqrt{5}$ and $|w| = \sqrt{4+a^2}$

$|w| = 2|z|$

$\sqrt{4+a^2} = 2\sqrt{5}$

attempt to solve equation ***M1***

Note: Award ***M0*** if modulus is not used.

$a = \pm 4$ ***A1A1*** ***N0***

(b) $zw = (2-2a) + (4+a)\mathrm{i}$ ***A1***

forming equation $2-2a = 2(4+a)$ ***M1***

$a = -\frac{3}{2}$ ***A1*** ***N0***

[6 marks]

2. (a) $-2 = 1 + k\sin\left(\frac{\pi}{6}\right)$ ***M1***

$-3 = \frac{1}{2}k$ ***A1***

$k = -6$ ***AG*** ***N0***

(b) **METHOD 1**

maximum $\Rightarrow \sin x = -1$ ***M1***

$a = \frac{3\pi}{2}$ ***A1***

$b = 1 - 6(-1)$

$= 7$ ***A1*** ***N2***

METHOD 2

$y' = 0$ ***M1***

$k\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$a = \frac{3\pi}{2}$ ***A1***

$b = 1 - 6(-1)$

$= 7$ ***A1*** ***N2***

Note: Award ***A1A1*** for $\left(\frac{3\pi}{2}, 7\right)$.

[5 marks]

3. $g(x) = 0$

$\log_5 |2\log_3 x| = 0$ ***(M1)***

$|2\log_3 x| = 1$ ***A1***

$\log_3 x = \pm\frac{1}{2}$ ***(A1)***

$x = 3^{\pm\frac{1}{2}}$ ***A1***

so the product of the zeros of g is $3^{\frac{1}{2}} \times 3^{-\frac{1}{2}} = 1$ ***A1*** ***N0***

[5 marks]

4. finding $\det \boldsymbol{A} = e^x - e^{-x}(2 + e^x)$ or equivalent ***A1***

$\boldsymbol{A}$ is singular $\Rightarrow \det \boldsymbol{A} = 0$ ***(R1)***

$e^x - e^{-x}(2 + e^x) = 0$

$e^{2x} - e^x - 2 = 0$ ***A1***

solving for e^x ***(M1)***

as $e^x > 0$ (or equivalent explanation) ***(R1)***

$e^x = 2$

$x = \ln 2$ (only) ***A1*** ***N0***

[6 marks]

5. (a) **METHOD 1**

let $x = \arctan\frac{1}{2} \Rightarrow \tan x = \frac{1}{2}$ and $y = \arctan\frac{1}{3} \Rightarrow \tan y = \frac{1}{3}$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = 1$ ***M1***

so, $x + y = \arctan 1 = \frac{\pi}{4}$ ***A1AG***

METHOD 2

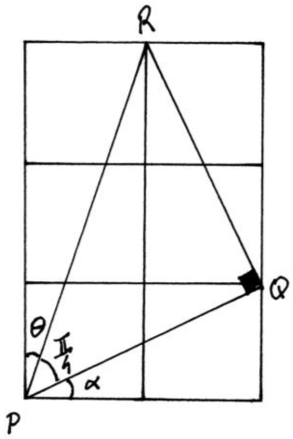
for $x, y > 0$, $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$ if $xy < 1$ ***M1***

so, $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right) = \frac{\pi}{4}$ ***A1AG***

METHOD 3

an appropriate sketch ***M1***

e.g.



correct reasoning leading to $\frac{\pi}{4}$ ***R1AG***

continued …

Question 5 continued

(b) **METHOD 1**

$$\arctan(2)+\arctan(3)=\frac{\pi}{2}-\arctan\left(\frac{1}{2}\right)+\frac{\pi}{2}-\arctan\left(\frac{1}{3}\right)$$ ***(M1)***

$$=\pi-\left(\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right)\right)$$ ***(A1)***

Note: Only one of the previous two marks may be implied.

$$=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$$ ***A1*** ***N1***

METHOD 2

let $x=\arctan 2\Rightarrow \tan x=2$ and $y=\arctan 3\Rightarrow \tan y=3$

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{2+3}{1-2\times 3}=-1$$ ***(M1)***

as $\frac{\pi}{4}<x<\frac{\pi}{2}$ $\left(\text{accept } 0<x<\frac{\pi}{2}\right)$

and $\frac{\pi}{4}<y<\frac{\pi}{2}$ $\left(\text{accept } 0<y<\frac{\pi}{2}\right)$

$\frac{\pi}{2}<x+y<\pi$ (accept $0<x+y<\pi$) ***(R1)***

Note: Only one of the previous two marks may be implied.

so, $x+y=\frac{3\pi}{4}$ ***A1*** ***N1***

METHOD 3

for $x,\ y>0$, $\arctan x+\arctan y=\arctan\left(\frac{x+y}{1-xy}\right)+\pi$ if $xy>1$ ***(M1)***

so, $\arctan 2+\arctan 3=\arctan\left(\frac{2+3}{1-2\times 3}\right)+\pi$ ***(A1)***

Note: Only one of the previous two marks may be implied.

$$=\frac{3\pi}{4}$$ ***A1*** ***N1***

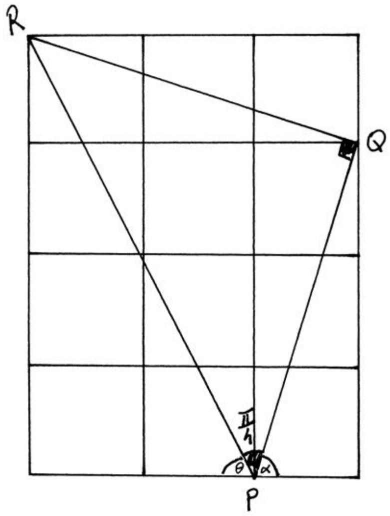
continued …

Question 5 continued

METHOD 4

an appropriate sketch ***M1***

e.g.



correct reasoning leading to $\frac{3\pi}{4}$ ***R1A1***

[5 marks]

6. $A = \frac{\theta}{2}(R^2 - r^2)$ ***A1***

$B = \frac{\theta}{2}r^2$ ***A1***

from $A : B = 2 : 1$, we have $R^2 - r^2 = 2r^2$ ***M1***

$R = \sqrt{3}r$ ***(A1)***

hence exact value of the ratio $R : r$ is $\sqrt{3} : 1$ ***A1*** ***N0***

[5 marks]

7. (a) $2^{\frac{1}{x}} = 4 - 2^{\frac{1}{x}}$

attempt to solve the equation ***M1***

$x = 1$ ***A1***

so P is $(1, 2)$, as $f(1) = 2$ ***A1*** ***N1***

(b) $f'(x) = -\frac{1}{x^2}2^{\frac{1}{x}}\ln 2$ ***A1***

attempt to substitute *x*-value found in part (a) into their $f'(x)$ ***M1***

$f'(1) = -2\ln 2$

$y - 2 = -2\ln 2(x - 1)$ (or equivalent) ***M1A1*** ***N0***

[7 marks]

8. **METHOD 1**

for finding two of the following three vectors (or their negatives)

$\vec{AB} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \vec{AC} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}, \vec{BC} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$ ***(A1)(A1)***

and calculating

EITHER

$\vec{AB} \times \vec{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 2 & -1 \\ -2 & 2 & -2 \end{vmatrix} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$ ***M1A1***

area $\Delta ABC = \frac{1}{2}\left| \vec{AB} \times \vec{AC} \right|$ ***M1***

OR

$\vec{BA} \times \vec{BC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & -2 & 1 \\ -2 & 0 & -1 \end{vmatrix} = \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix}$ ***M1A1***

area $\Delta ABC = \frac{1}{2}\left| \vec{BA} \times \vec{BC} \right|$ ***M1***

OR

$\vec{CA} \times \vec{CB} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -2 & 2 \\ 2 & 0 & 1 \end{vmatrix} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$ ***M1A1***

area $\Delta ABC = \frac{1}{2}\left| \vec{CA} \times \vec{CB} \right|$ ***M1***

THEN

area $\Delta ABC = \frac{\sqrt{24}}{2}$ ***A1***

$= \sqrt{6}$ ***AG*** ***N0***

continued …

Question 8 continued

METHOD 2

for finding two of the following three vectors (or their negatives)

$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}, \overrightarrow{BC} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$ ***(A1)(A1)***

EITHER

$\cos A = \dfrac{\overrightarrow{AB} \bullet \overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}$ ***M1***

$= \dfrac{6}{\sqrt{5}\sqrt{12}} = \dfrac{6}{\sqrt{60}} \quad \left(\text{or } \dfrac{3}{\sqrt{15}}\right)$

$\sin A = \sqrt{\dfrac{2}{5}}$ ***A1***

$\text{area } \Delta\text{ABC} = \dfrac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\sin A$ ***M1***

$= \dfrac{1}{2}\sqrt{5}\sqrt{12}\sqrt{\dfrac{2}{5}}$

$= \dfrac{1}{2}\sqrt{24}$ ***A1***

$= \sqrt{6}$ ***AG*** ***N0***

OR

$\cos B = \dfrac{\overrightarrow{BA} \bullet \overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|}$ ***M1***

$= -\dfrac{1}{\sqrt{5}\sqrt{5}} = -\dfrac{1}{5}$

$\sin B = \sqrt{\dfrac{24}{25}} \quad \left(\text{or } \dfrac{\sqrt{24}}{5}\right)$ ***A1***

$\text{area } \Delta\text{ABC} = \dfrac{1}{2}\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|\sin B$ ***M1***

$= \dfrac{1}{2}\sqrt{5}\sqrt{5}\sqrt{\dfrac{24}{25}}$

$= \dfrac{1}{2}\sqrt{24}$ ***A1***

$= \sqrt{6}$ ***AG*** ***N0***

continued …

Question 8 continued

OR

$\cos C = \dfrac{\vec{CA} \bullet \vec{CB}}{\left|\vec{CA}\right|\left|\vec{CB}\right|}$ ***M1***

$= \dfrac{6}{\sqrt{12}\sqrt{5}} = \dfrac{6}{\sqrt{60}} \left(\text{or } \dfrac{3}{\sqrt{15}}\right)$

$\sin C = \sqrt{\dfrac{2}{5}}$ ***A1***

area $\Delta\text{ABC} = \dfrac{1}{2}\left|\vec{CA}\right|\left|\vec{CB}\right|\sin C$ ***M1***

$= \dfrac{1}{2}\sqrt{12}\sqrt{5}\sqrt{\dfrac{2}{5}}$

$= \dfrac{1}{2}\sqrt{24}$ ***A1***

$= \sqrt{6}$ ***AG*** ***N0***

METHOD 3

for finding two of the following three vectors (or their negatives)

$\vec{AB} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \vec{AC} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}, \vec{BC} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$ ***(A1)(A1)***

$\text{AB} = \sqrt{5} = c$, $\text{AC} = \sqrt{12} = 2\sqrt{3} = b$, $\text{BC} = \sqrt{5} = a$ ***M1A1***

$s = \dfrac{\sqrt{5} + 2\sqrt{3} + \sqrt{5}}{2} = \sqrt{3} + \sqrt{5}$ ***M1***

area $\Delta\text{ABC} = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{(\sqrt{3}+\sqrt{5})(\sqrt{3})(\sqrt{5}-\sqrt{3})(\sqrt{3})}$

$= \sqrt{3(5-3)}$ ***A1***

$= \sqrt{6}$ ***AG*** ***N0***

continued …

Question 8 continued

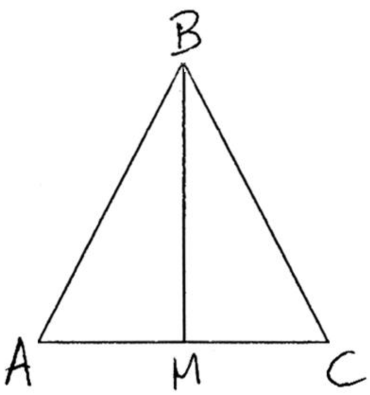
METHOD 4

for finding two of the following three vectors (or their negatives)

$\vec{AB} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \vec{AC} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}, \vec{BC} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$ ***(A1)(A1)***

$AB = BC = \sqrt{5}$ and $AC = \sqrt{12} = 2\sqrt{3}$ ***M1A1***

ΔABC is isosceles



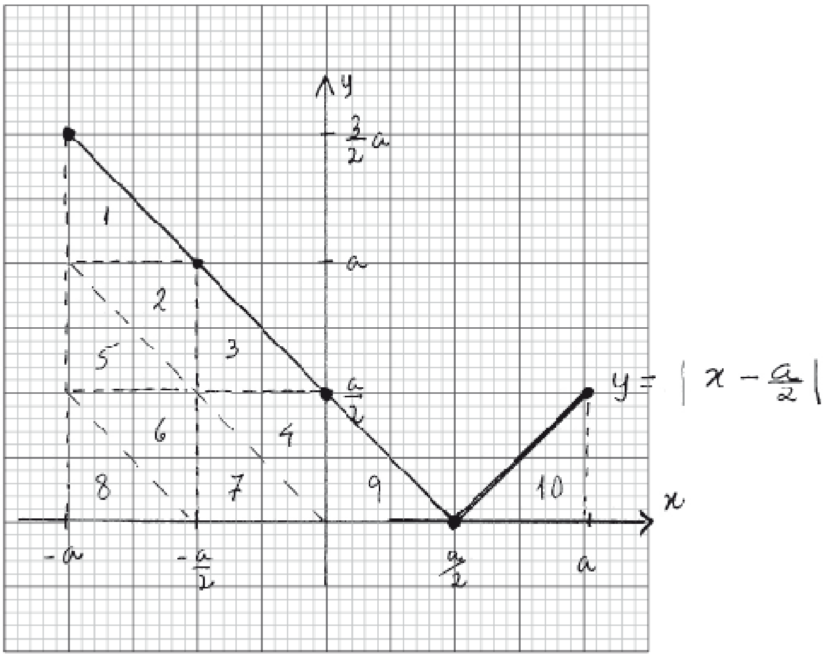
let M be the midpoint of $[AC]$, the height $BM = \sqrt{5-3} = \sqrt{2}$ ***M1***

area $\Delta ABC = \frac{2\sqrt{3} \times \sqrt{2}}{2}$ ***A1***

$= \sqrt{6}$ ***AG*** ***N0***

[6 marks]

9. (a)



A1A1

> **Note:** Award ***A1*** for the correct x-intercept, ***A1*** for completely correct graph.

(b) **METHOD 1**

the area under the graph of $y = \left|x - \frac{a}{2}\right|$ for $-a \le x \le a$, can be divided into ten congruent triangles; ***M1A1***

the area of eight of these triangles is given by $\int_{-a}^{0} \left|x - \frac{a}{2}\right| dx$ and the areas of the other two by $\int_{0}^{a} \left|x - \frac{a}{2}\right| dx$ ***M1A1***

so, $\int_{-a}^{0} \left|x - \frac{a}{2}\right| dx = 4\int_{0}^{a} \left|x - \frac{a}{2}\right| dx \Rightarrow k = 4$ ***A1*** ***N0***

METHOD 2

use area of trapezium to calculate ***M1***

$\int_{-a}^{0} \left|x - \frac{a}{2}\right| dx = a \times \frac{1}{2}\left(\frac{3a}{2} + \frac{a}{2}\right) = a^2$ ***A1***

and area of two triangles to obtain ***M1***

$\int_{0}^{a} \left|x - \frac{a}{2}\right| dx = 2 \times \frac{1}{2}\left(\frac{a}{2}\right)^2 = \frac{a^2}{4}$ ***A1***

so, $k = 4$ ***A1*** ***N0***

continued ...

Question 9 continued

METHOD 3

use integration to find the area under the curve

$$\int_{-a}^{0}\left|x-\frac{a}{2}\right|\mathrm{d}x=\int_{-a}^{0}-x+\frac{a}{2}\mathrm{d}x$$ ***M1***

$$=\left[-\frac{x^2}{2}+\frac{a}{2}x\right]_{-a}^{0}=\frac{a^2}{2}+\frac{a^2}{2}=a^2$$ ***A1***

and

$$\int_{0}^{a}\left|x-\frac{a}{2}\right|\mathrm{d}x=\int_{0}^{\frac{a}{2}}-x+\frac{a}{2}\mathrm{d}x+\int_{\frac{a}{2}}^{a}x-\frac{a}{2}\mathrm{d}x$$ ***M1***

$$=\left[-\frac{x^2}{2}+\frac{a}{2}x\right]_{0}^{\frac{a}{2}}+\left[\frac{x^2}{2}-\frac{a}{2}x\right]_{\frac{a}{2}}^{a}=-\frac{a^2}{8}+\frac{a^2}{4}+\frac{a^2}{2}-\frac{a^2}{2}-\frac{a^2}{8}+\frac{a^2}{4}=\frac{a^2}{4}$$ ***A1***

so, $k=4$ ***A1*** ***N0***

[7 marks]

10. (a) **METHOD 1**

$V = a^3 - \frac{1}{a^3}$ ***A1***

$x^3 = \left(a - \frac{1}{a}\right)^3$ ***M1***

$= a^3 - 3a + \frac{3}{a} - \frac{1}{a^3}$

$= a^3 - \frac{1}{a^3} - 3\left(a - \frac{1}{a}\right)$ (or equivalent) ***(A1)***

$\Rightarrow a^3 - \frac{1}{a^3} = x^3 + 3x$

$V = x^3 + 3x$ ***A1*** ***N0***

METHOD 2

$V = a^3 - \frac{1}{a^3}$ ***A1***

attempt to use difference of cubes formula, $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$ ***M1***

$V = \left(a - \frac{1}{a}\right)\left(a^2 + 1 + \left(\frac{1}{a}\right)^2\right)$

$= \left(a - \frac{1}{a}\right)\left(\left(a - \frac{1}{a}\right)^2 + 3\right)$ ***(A1)***

$= x(x^2 + 3)$ or $x^3 + 3x$ ***A1*** ***N0***

METHOD 3

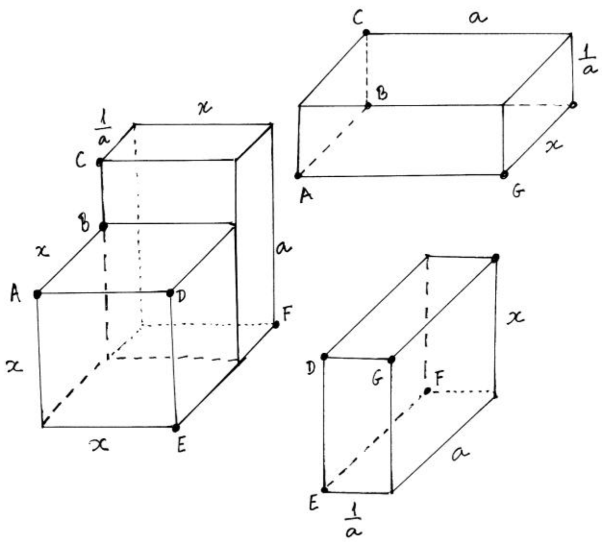


diagram showing that the solid can be decomposed ***M1***

into three congruent $x \times a \times \frac{1}{a}$ cuboids with volume x ***A1***

and a cube with edge x with volume x^3 ***A1***

so, $V = x^3 + 3x$ ***A1*** ***N0***

continued ...

Question 10 continued

(b)

Note:	Do not accept any method where candidate substitutes the given value of a into $x = a - \frac{1}{a}$.

METHOD 1

$V = 4x \Leftrightarrow x^3 + 3x = 4x \Leftrightarrow x^3 - x = 0$ ***M1***

$\Leftrightarrow x(x-1)(x+1) = 0$

$\Rightarrow x = 1$ as $x > 0$ ***A1***

so, $a - \frac{1}{a} = 1 \Rightarrow a^2 - a - 1 = 0 \Rightarrow a = \frac{1 \pm \sqrt{5}}{2}$ ***M1A1***

as $a > 1$, $a = \frac{1+\sqrt{5}}{2}$ ***AG*** ***N0***

METHOD 2

$a^3 - \frac{1}{a^3} = 4\left(a - \frac{1}{a}\right) \Rightarrow a^6 - 4a^4 + 4a^2 - 1 = 0 \Leftrightarrow (a^2 - 1)(a^4 - 3a^2 + 1) = 0$ ***M1A1***

as $a > 1 \Rightarrow a^2 > 1$, $a^2 = \frac{3+\sqrt{5}}{2} \Leftrightarrow a^2 = \sqrt{\left(\frac{1+\sqrt{5}}{2}\right)^2}$ ***M1A1***

$\Rightarrow a = \frac{1+\sqrt{5}}{2}$ ***AG*** ***N0***

[8 marks]

SECTION B

11. (a) $f(1) = 1 - \arctan 1 = 1 - \frac{\pi}{4}$ ***A1***

$f(-\sqrt{3}) = -\sqrt{3} - \arctan(-\sqrt{3}) = -\sqrt{3} + \frac{\pi}{3}$ ***A1***

[2 marks]

(b) $f(-x) = -x - \arctan(-x)$ ***M1***

$= -x + \arctan x$ ***A1***

$= -(x - \arctan x)$

$= -f(x)$ ***AG*** ***N0***

[2 marks]

(c) as $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$, for any $x \in \mathbb{R}$ ***A1***

$\Rightarrow \ -\frac{\pi}{2} < -\arctan x < \frac{\pi}{2}$, for any $x \in \mathbb{R}$

then by adding x (or equivalent) ***R1***

we have $x - \frac{\pi}{2} < x - \arctan x < x + \frac{\pi}{2}$ ***AG*** ***N0***

[2 marks]

(d) $f'(x) = 1 - \frac{1}{1+x^2}$ or $\frac{x^2}{1+x^2}$ ***A1A1***

$f''(x) = \frac{2x(1+x^2) - 2x^3}{(1+x^2)^2}$ or $\frac{2x}{(1+x^2)^2}$ ***M1A1***

$f'(0) = f''(0) = 0$ ***A1A1***

EITHER

as $f'(x) \geq 0$ for all values of $x \in \mathbb{R}$
($(0, 0)$ is not an extreme of the graph of f (or equivalent)) ***R1***

OR

as $f''(x) > 0$ for positive values of x and $f''(x) < 0$ for negative values of x ***R1***

THEN

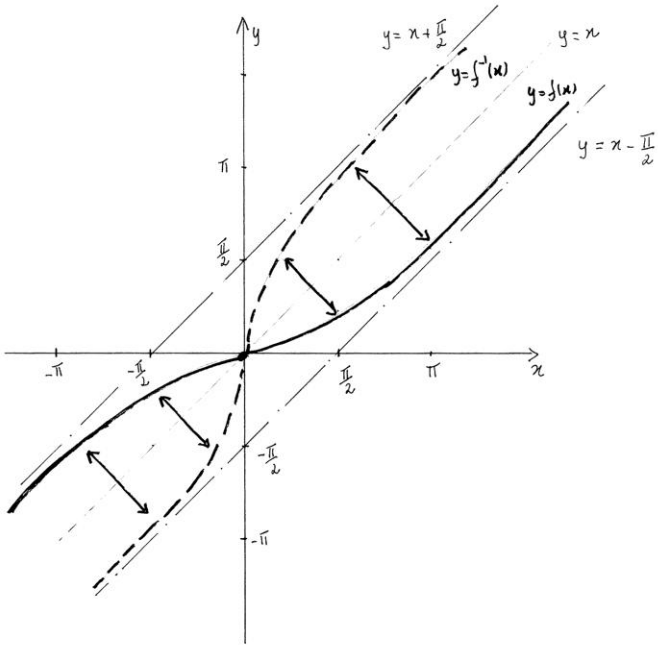
$(0, 0)$ is a point of inflexion of the graph of f (with zero gradient) ***A1*** ***N2***

[8 marks]

continued …

Question 11 continued

(e)



A1A1A1

Note: Award ***A1*** for both asymptotes.	
A1 for correct shape (concavities) $x < 0$.	
A1 for correct shape (concavities) $x > 0$.	

[3 marks]

(f) (see sketch above)

as f is increasing (and therefore one-to-one) and its range is $\mathbb{R}$,

f^{-1} is defined for all $x \in \mathbb{R}$ ***R1***

use the result that the graph of $y = f^{-1}(x)$ is the reflection

in the line $y = x$ of the graph of $y = f(x)$ to draw the graph of f^{-1} ***(M1)A1***

[3 marks]

Total [20 marks]

12. (a) (i) $a, 2a, 3a, ..., na$ are n consecutive terms of an AP with first term a and common difference a

so their mean is $\frac{a+2a+3a+...+na}{n} = \frac{a\frac{n(n+1)}{2}}{n}$ ***M1A1***

$= \frac{a(n+1)}{2}$ ***AG*** ***N0***

(ii) $4+2\times 4+3\times 4+...+4n > \frac{4(n+1)}{2}+100$ ***M1***

$\frac{4n(n+1)}{2} > 2(n+1)+100$ ***A1***

$2n^2+2n > 2n+102$

attempt to solve ***(M1)***

$n^2 > 51$

so the minimum value of n that satisfies the condition is 8 ***A1*** ***N0***

Note:	Award ***M1A1(M1)A1*** for use of equations if there is a clear conversion to an inequality.

.

[6 marks]

(b) (i) $M = \frac{x_1+...+x_m+y_1+...+y_n}{m+n}$ ***M1***

$= \frac{0\times m+1\times n}{m+n}$ ***A1***

$= \frac{n}{m+n}$ ***AG*** ***N0***

EITHER

$S = \sqrt{\frac{\left(0-\frac{n}{m+n}\right)^2\times m+\left(1-\frac{n}{m+n}\right)^2\times n}{m+n}}$ ***M1A1***

attempt to simplify ***M1***

$S = \sqrt{\frac{\frac{m^2n+n^2m}{(m+n)^2}}{m+n}} = \sqrt{\frac{mn(m+n)}{(m+n)^3}}$

$= \sqrt{\frac{mn}{(m+n)^2}}$ ***A1***

$= \frac{\sqrt{mn}}{m+n}$ ***AG*** ***N0***

continued …

Question 12(b)(i) continued

OR

$\text{Var}(x) = \dfrac{\sum_{i=1}^{m} x_i^2 + \sum_{i=1}^{n} y_i^2}{m+n} - M^2$ ***M1A1***

attempt to simplify ***M1***

$\text{Var}(x) = \dfrac{n}{m+n} - \dfrac{n^2}{(m+n)^2}$

$= \dfrac{n}{m+n}\left(1 - \dfrac{n}{m+n}\right)$

$= \dfrac{n}{m+n} \times \dfrac{m}{m+n}$

$= \dfrac{mn}{(m+n)^2}$ ***A1***

$\therefore S = \dfrac{\sqrt{mn}}{m+n}$ ***AG*** ***N0***

(ii) $M = S \Rightarrow \dfrac{n}{m+n} = \dfrac{\sqrt{mn}}{m+n}$ ***A1***

attempt to solve ***M1***

$\Rightarrow n = \sqrt{mn}$

$\Rightarrow n = m$, as $n > 0$ ***A1***

so, then the set has $2n$ numbers, $x_1, \ldots, x_n, y_1, \ldots, y_n$

from which the first n are all 0 and the last n are all 1 ***(M1)***

hence the value of the median is $\dfrac{x_n + y_1}{2} = \dfrac{1}{2}$ ***A1*** ***N0***

[11 marks]

Total [17 marks]

13. Part A

(a) $|z| = z$, $\arg(z) = 0$ ***A1A1***
so $L(z) = \ln z$ ***AG*** ***N0***
[2 marks]

(b) (i) $L(-1) = \ln 1 + \mathrm{i}\pi = \mathrm{i}\pi$ ***A1A1*** ***N2***

(ii) $L(1-\mathrm{i}) = \ln\sqrt{2} + \mathrm{i}\frac{7\pi}{4}$ ***A1A1*** ***N2***

(iii) $L(-1+\mathrm{i}) = \ln\sqrt{2} + \mathrm{i}\frac{3\pi}{4}$ ***A1*** ***N1***
[5 marks]

(c) for comparing the product of two of the above results with the third ***M1***
for stating the result $-1+\mathrm{i} = -1\times(1-\mathrm{i})$ and $L(-1+\mathrm{i}) \neq L(-1) + L(1-\mathrm{i})$ ***R1***
hence, the property $L(z_1 z_2) = L(z_1) + L(z_2)$
does not hold for all values of z_1 and z_2 ***AG*** ***N0***
[2 marks]

Sub-total [9 marks]

continued …

Question 13 continued

Part B

(a) from $f(x+y)=f(x)f(y)$

for $x=y=0$ ***M1***

we have $f(0+0)=f(0)f(0)\Leftrightarrow f(0)=\left(f(0)\right)^2$ ***A1***

as $f(0)\neq 0$, this implies that $f(0)=1$ ***R1AG*** ***N0***

[3 marks]

(b) **METHOD 1**

from $f(x+y)=f(x)f(y)$

for $y=-x$, we have $f(x-x)=f(x)f(-x)\Leftrightarrow f(0)=f(x)f(-x)$ ***M1A1***

as $f(0)\neq 0$ this implies that $f(x)\neq 0$ ***R1AG*** ***N0***

METHOD 2

suppose that, for a value of *x*, $f(x)=0$ ***M1***

from $f(x+y)=f(x)f(y)$

for $y=-x$, we have $f(x-x)=f(x)f(-x)\Leftrightarrow f(0)=f(x)f(-x)$ ***A1***

substituting $f(x)$ by 0 gives $f(0)=0$ which contradicts part (a) ***R1***

therefore $f(x)\neq 0$ for all *x*. ***AG*** ***N0***

[3 marks]

(c) by the definition of derivative

$$f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$ ***(M1)***

$$=\lim_{h\to 0}\left(\frac{f(x)f(h)-f(x)f(0)}{h}\right)$$ ***A1(A1)***

$$=\lim_{h\to 0}\left(\frac{f(h)-f(0)}{h}\right)f(x)$$ ***A1***

$$=f'(0)f(x)\quad\left(=k\,f(x)\right)$$ ***AG*** ***N0***

[4 marks]

(d) $\int\frac{f'(x)}{f(x)}\,\mathrm{d}x=\int k\,\mathrm{d}x\Rightarrow\ln f(x)=kx+C$ ***M1A1***

$\ln f(0)=C\Rightarrow C=0$ ***A1***

$f(x)=\mathrm{e}^{kx}$ ***A1*** ***N1***

Note:	Award ***M1A0A0A0*** if no arbitrary constant C.

[4 marks]

Sub-total [14 marks]

Total [23 marks]