M08/5/MATHL/HP3/ENG/TZ1/SG

22087211

International Baccalaureate®
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MATHEMATICS
HIGHER LEVEL
PAPER 3 – SETS, RELATIONS AND GROUPS

Monday 19 May 2008 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 12]*

The binary operation $*$ is defined for $a, b \in \mathbb{Z}^+$ by

$$a * b = a + b - 2 .$$

(a) Determine whether or not $*$ is

(i) closed,

(ii) commutative,

(iii) associative. *[7 marks]*

(b) (i) Find the identity element.

(ii) Find the set of positive integers having an inverse under $*$. *[5 marks]*

2. *[Maximum mark: 10]*

The function f is defined by

$$f(x) = \frac{1 - e^{-x}}{1 + e^{-x}},\ x \in \mathbb{R} .$$

(a) Find the range of f. *[2 marks]*

(b) Prove that f is an injection. *[3 marks]*

(c) Taking the codomain of f to be equal to the range of f, find an expression for $f^{-1}(x)$. *[5 marks]*

3. *[Maximum mark: 17]*

(a) Find the six roots of the equation $z^6 - 1 = 0$, giving your answers in the form $r \operatorname{cis} \theta$, $r \in \mathbb{R}^+$, $0 \le \theta < 2\pi$. *[4 marks]*

(b) (i) Show that these six roots form a group G under multiplication of complex numbers.

(ii) Show that G is cyclic and find all the generators.

(iii) Give an example of another group that is isomorphic to G, stating clearly the corresponding elements in the two groups. *[13 marks]*

4. *[Maximum mark: 9]*

The relation R is defined on ordered pairs by

$$(a, b) R (c, d) \text{ if and only if } ad = bc \text{ where } a, b, c, d \in \mathbb{R}^+.$$

(a) Show that R is an equivalence relation. *[6 marks]*

(b) Describe, geometrically, the equivalence classes. *[3 marks]*

5. *[Maximum mark: 12]*

Let $p = 2^k + 1$, $k \in \mathbb{Z}^+$ be a prime number and let G be the group of integers $1, 2, \ldots, p - 1$ under multiplication defined modulo p.

By first considering the elements $2^1, 2^2, \ldots, 2^k$ and then the elements $2^{k+1}, 2^{k+2}, \ldots$, show that the order of the element 2 is $2k$.

Deduce that $k = 2^n$ for $n \in \mathbb{N}$.