22087213

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 1

Wednesday 7 May 2008 (afternoon)

2 hours

Candidate session number

0	0							

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** the questions in the spaces provided. Working may be continued below the lines, if necessary.*

1. *[Maximum mark: 6]*

The probability distribution of a discrete random variable X is defined by

$$P(X = x) = cx(5 - x), \ x = 1, 2, 3, 4 .$$

(a) Find the value of c. *[3 marks]*

(b) Find $E(X)$. *[3 marks]*

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2. *[Maximum mark: 6]*

The polynomial $P(x) = x^3 + ax^2 + bx + 2$ is divisible by $(x+1)$ and by $(x-2)$.

Find the value of a and of b, where $a, b \in \mathbb{R}$.

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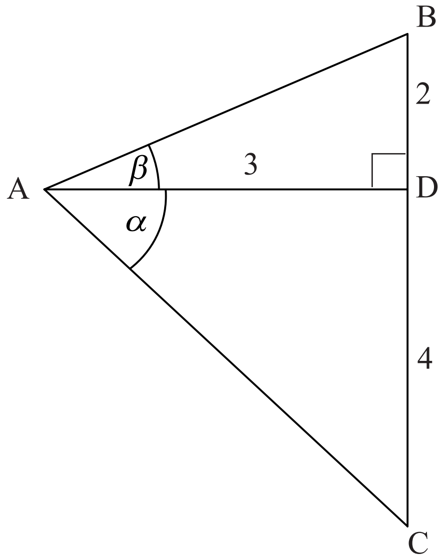
3. *[Maximum mark: 6]*

In the diagram below, AD is perpendicular to BC.
$\text{CD} = 4$, $\text{BD} = 2$ and $\text{AD} = 3$. $\text{C}\hat{\text{A}}\text{D} = \alpha$ and $\text{B}\hat{\text{A}}\text{D} = \beta$.



Find the exact value of $\cos(\alpha - \beta)$.

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4. *[Maximum mark: 6]*

Let $f(x) = \frac{4}{x+2}$, $x \neq -2$ and $g(x) = x - 1$.

If $h = g \circ f$, find

(a) $h(x)$; *[2 marks]*

(b) $h^{-1}(x)$, where h^{-1} is the inverse of h. *[4 marks]*

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5. *[Maximum mark: 6]*

Consider the curve with equation $x^2 + xy + y^2 = 3$.

(a) Find in terms of k, the gradient of the curve at the point $(-1, k)$. *[5 marks]*

(b) Given that the tangent to the curve is parallel to the x-axis at this point, find the value of k. *[1 mark]*

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6. *[Maximum mark: 6]*

Show that $\int_0^{\frac{\pi}{6}} x \sin 2x \, dx = \frac{\sqrt{3}}{8} - \frac{\pi}{24}$.

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7. *[Maximum mark: 6]*

Let A and B be events such that $\mathrm{P}(A) = 0.6$, $\mathrm{P}(A \cup B) = 0.8$ and $\mathrm{P}(A \mid B) = 0.6$.

Find $\mathrm{P}(B)$.

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8. *[Maximum mark: 6]*

A normal to the graph of $y = \arctan(x-1)$, for $x > 0$, has equation $y = -2x + c$, where $c \in \mathbb{R}$.

Find the value of c.

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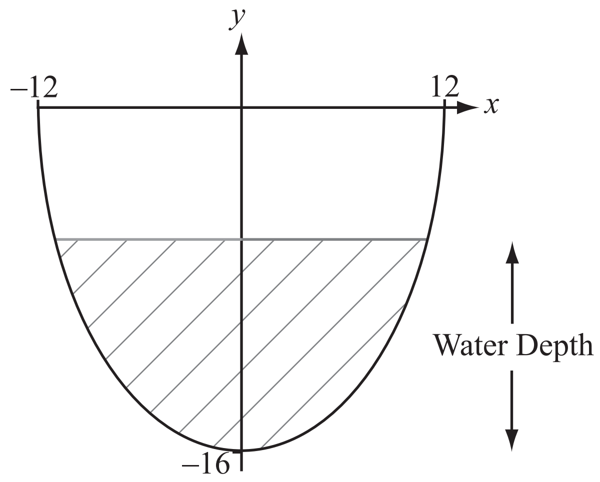
9. *[Maximum mark: 6]*

The diagram below shows the boundary of the cross-section of a water channel.



The equation that represents this boundary is $y = 16\sec\left(\frac{\pi x}{36}\right) - 32$ where x and y are both measured in cm.

The top of the channel is level with the ground and has a width of 24 cm. The maximum depth of the channel is 16 cm.

Find the width of the water surface in the channel when the water depth is 10 cm. Give your answer in the form $a \arccos b$ where $a, b \in \mathbb{R}$.

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10. *[Maximum mark: 6]*

Given any two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, show that $\left|\boldsymbol{a}\times\boldsymbol{b}\right|^2 = \left|\boldsymbol{a}\right|^2\left|\boldsymbol{b}\right|^2 - (\boldsymbol{a}\bullet\boldsymbol{b})^2$.

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SECTION B

*Answer **all** the questions on the answer sheets provided. Please start each question on a new page.*

11. *[Maximum mark: 20]*

Consider the points A$(1, -1, 4)$, B$(2, -2, 5)$ and O$(0, 0, 0)$.

(a) Calculate the cosine of the angle between $\overrightarrow{\text{OA}}$ and $\overrightarrow{\text{AB}}$. *[5 marks]*

(b) Find a vector equation of the line L_1 which passes through A and B. *[2 marks]*

The line L_2 has equation $\boldsymbol{r} = 2\boldsymbol{i} + 4\boldsymbol{j} + 7\boldsymbol{k} + t(2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})$, where $t \in \mathbb{R}$.

(c) Show that the lines L_1 and L_2 intersect and find the coordinates of their point of intersection. *[7 marks]*

(d) Find the Cartesian equation of the plane which contains both the line L_2 and the point A. *[6 marks]*

12. *[Maximum mark: 10]*

(a) Find the sum of the infinite geometric sequence $27, -9, 3, -1, \ldots$. *[3 marks]*

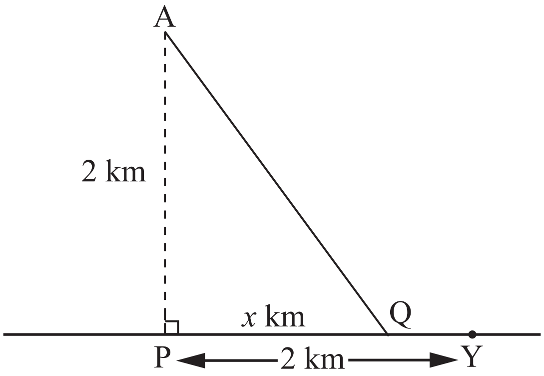
(b) Use mathematical induction to prove that for $n \in \mathbb{Z}^+$,

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

[7 marks]

13. *[Maximum mark: 18]*

André wants to get from point A located in the sea to point Y located on a straight stretch of beach. P is the point on the beach nearest to A such that $AP = 2\text{ km}$ and $PY = 2\text{ km}$. He does this by swimming in a straight line to a point Q located on the beach and then running to Y.



When André swims he covers 1 km in $5\sqrt{5}$ minutes. When he runs he covers 1 km in 5 minutes.

(a) If $PQ = x\text{ km}$, $0 \le x \le 2$, find an expression for the time T minutes taken by André to reach point Y. *[4 marks]*

(b) Show that $\dfrac{\mathrm{d}T}{\mathrm{d}x} = \dfrac{5\sqrt{5}\,x}{\sqrt{x^2+4}} - 5$. *[3 marks]*

(c) (i) Solve $\dfrac{\mathrm{d}T}{\mathrm{d}x} = 0$.

(ii) Use the value of x found in **part (c) (i)** to determine the time, T minutes, taken for André to reach point Y.

(iii) Show that $\dfrac{\mathrm{d}^2T}{\mathrm{d}x^2} = \dfrac{20\sqrt{5}}{(x^2+4)^{\frac{3}{2}}}$ and **hence** show that the time found in **part (c) (ii)** is a minimum. *[11 marks]*

14. *[Maximum mark: 12]*

Let $w = \cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}$.

(a) Show that w is a root of the equation $z^5 - 1 = 0$. *[3 marks]*

(b) Show that $(w-1)(w^4 + w^3 + w^2 + w + 1) = w^5 - 1$ and deduce that $w^4 + w^3 + w^2 + w + 1 = 0$. *[3 marks]*

(c) **Hence** show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$. *[6 marks]*
