IB DIPLOMA PROGRAMME
PROGRAMME DU DIPLÔME DU BI
PROGRAMA DEL DIPLOMA DEL BI

N07/5/MATHL/HP3/ENG/TZ0/XX/M+

MARKSCHEME

November 2007

MATHEMATICS

Higher Level

Paper 3

26 pages

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Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**: often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question. Award the marks as usual and then write –1(****MR****) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled ***(d)*** *and a brief* ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \ 2\cos(5x-3)\ \ 5 \quad = 10\cos(5x-3)$ ***A1***

  Award ***A1*** for $\ 2\cos(5x-3)\ \ 5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

*Candidates should be penalized **once only IN THE PAPER** for an accuracy error **(AP)**. Award the marks as usual then write **(AP)** against the answer. On the **front** cover write –1**(AP)**. Deduct 1 mark from the total for the paper, not the question*.

- If a final correct answer is incorrectly rounded, apply the ***AP***.
- If the level of accuracy is not specified in the question, apply the ***AP*** for correct answers not given to three significant figures.

If there is no working shown, and answers are given to the correct two significant figures, apply the ***AP***. However, do **not** accept answers to one significant figure without working.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Examples

Exemplar material is available under examiner training on http://courses.triplealearning.co.uk. Please refer to this material before you start marking, and when you have any queries. Please also feel free to contact your Team Leader if you need further advice.

SECTION A

Statistics and probability

1. (a) $\mathrm{P}(X < 57) = \mathrm{P}\left(Z < \frac{57-75}{12}\right)$ ***(M1)***

$= \mathrm{P}(Z < -1.50) = 0.0668$ ***A1***

(same value from tables)

[2 marks]

(b) $\mathrm{P}\left(Z < \left(\frac{50-45}{\sigma}\right)\right) = 0.7$ ***(M1)***

$\frac{50-45}{\sigma} = 0.5244$ ***A1***

$\sigma = \frac{50-45}{0.5244}$

$= 9.53$ ***A1***

[3 marks]

(c) $\mathrm{H}_0 : \mu_{current\,affairs} = 75;\quad \mathrm{H}_1 : \mu_{current\,affairs} > 75$ ***A1***

By GDC for the sample $\bar{x} = 83.7, s_x = 7.08754...$ ***(A1)***

for small sample with $n = 10$,

$t = \frac{83.7-75}{\frac{7.08754...}{\sqrt{10}}} = 3.8817$ ***(M1)(A1)***

EITHER

critical value at the 5 % level $\nu = 9$ is 1.833. ***A1***

$3.8817 > 1.833$ hence reject H_0 and accept H_1 ***R1***

OR

p-value $= 0.00186$ so reject H_0 since $0.00186 < 0.05$ ***A1R1***

[6 marks]

Total [11 marks]

2. (a) **METHOD 1**

H_0 : distribution is $B(6, 0.5)$; H_1 : distribution is not $B(6, 0.5)$ ***A1***

	0	1	2	3	4	5	6
Observed frequency	1	5	26	37	18	12	1
Expected frequency	$\frac{25}{16} = 1.5625$	$\frac{150}{16} = 9.375$	$\frac{375}{16} = 23.4375$	$\frac{500}{16} = 31.25$	$\frac{375}{16} = 23.4375$	$\frac{150}{16} = 9.375$	$\frac{25}{16} = 1.5625$

$$\left(E_0 = 100(0.5)^6 = \frac{25}{16} = 0.015625\right)$$ ***A3***

Combining the first two columns and the last two columns: ***A1***

$$\chi^2 = \sum \frac{O^2}{E} - \sum E$$

$$= \frac{6^2}{\left(\frac{175}{16}\right)} + \frac{26^2}{\left(\frac{375}{16}\right)} + \frac{37^2}{\left(\frac{500}{16}\right)} + \frac{18^2}{\left(\frac{375}{16}\right)} + \frac{13^2}{\left(\frac{175}{16}\right)} - 100$$ ***(M1)***

$$= 5.22$$ ***A1***

$\nu = 4$, so critical value of $\chi^2_{5\%} = 9.488$ ***A1A1***

Since $5.22 < 9.488$ the result is not significant and we accept H_0 ***R1***

[10 marks]

METHOD 2

H_0 : distribution is $B(6, 0.5)$; H_1 : distribution is not $B(6, 0.5)$ ***A1***

By GDC, $p = 0.266$ ***A8***

Since $0.266 > 0.05$ the result is not significant and we accept H_0. ***R1***

[10 marks]

(b) Estimate *p* from the data which would entail the loss of one degree of freedom ***A1A1***

[2 marks]

continued …

Question 2 continued

(c) **METHOD 1**

H_0 : there is no association H_1 : there is an association ***A1***

Outcome	O	**Day** E	O	**Night** E	
Live males	68	64.8	42	45.2	110
Live females	103	94.3	57	65.7	160
Dead males	8	15.3	18	10.7	26
Dead females	6	10.6	12	7.4	18
	185		129		314

A2

$$\chi^2 = \frac{68^2}{64.8} + \frac{42^2}{45.2} + \frac{103^2}{94.3} + \ldots + \frac{6^2}{10.6} + \frac{12^2}{7.4} - 314$$ ***(M1)***

$= 15.7$ ***A1***

$\nu = 3$, $\chi^2_{5\%}(3) = 7.815$ ***A1A1***

Since $15.7 > 7.815$ we reject H_0 ***R1***

[8 marks]

METHOD 2

H_0 : there is no association H_1 : there is an association ***A1***

By GDC, $p = 0.00129$ ***A6***

Since $0.00129 < 0.05$ we reject H_0. ***R1***

[8 marks]

Total [20 marks]

3. (a) **EITHER**

Median, m satisfies

$\int_0^m \lambda e^{-\lambda x} dx = \left[-e^{-\lambda x}\right]_0^m = \frac{1}{2}$ ***M1A1***

$1 - e^{-\lambda m} = \frac{1}{2}$

$e^{-\lambda m} = \frac{1}{2}$ ***A1***

$e^{\lambda m} = 2$

$\lambda m = \ln 2 \rightarrow m = \frac{\ln 2}{\lambda}$; mean is $\frac{1}{\lambda}$ ***A1***

Hence mean > median ***AG***

OR

$P\left(X < \frac{1}{\lambda}\right) = \int_0^{\frac{1}{\lambda}} \lambda e^{-\lambda x} dx$ ***M1A1***

$= \left[-e^{-\lambda x}\right]_0^{\frac{1}{\lambda}}$ ***A1***

$= 1 - e^{-1} = 0.6321$ ***A1***

Hence mean > median ***AG***

[4 marks]

(b) (i) $f(x) = 0.1e^{-0.1x}$

$P(X > 20) = \int_{20}^{\infty} 0.1e^{-0.1x} dx = \left[-e^{-0.1x}\right]_{20}^{\infty}$ ***M1A1***

$= e^{-2}$

$= 0.135$ ***A1***

continued ...

Question 3 (b) continued

(ii) **EITHER**

$\mathrm{P}(\text{next butterfly within 50 seconds of first}) = 1 - \mathrm{e}^{-(0.1)\times(50-20)}$ ***M1A1***

$= 1 - \mathrm{e}^{-3}$

$= 0.950$ ***A1***

OR

Using the memoryless property,

$\mathrm{P}(T \le 50 \mid T > 20) = \mathrm{P}(0 < T \le 30)$ ***M1***

$= \int_0^{30} 0.1\mathrm{e}^{-0.1t}\,\mathrm{d}t$ ***A1***

$= 1 - \mathrm{e}^{-3} = 0.950$ ***A1***

OR

$\mathrm{P}(T \le 50 \mid T > 20) = \dfrac{\mathrm{P}(20 < T \le 50)}{\mathrm{P}(T > 20)}$ ***M1***

$= \dfrac{\int_{20}^{50} 0.1\mathrm{e}^{-0.1t}\,\mathrm{d}t}{\mathrm{e}^{-2}}$ ***A1***

$= \dfrac{\mathrm{e}^{-2} - \mathrm{e}^{-5}}{\mathrm{e}^{-2}} = 0.950$ ***A1***

[6 marks]

(c) (i) $\mathrm{e}^{\frac{-t}{36}}$ ***A1***

(ii) $1 - F(t) = \mathrm{P}(T > t)$ ***M1***

$= \mathrm{P}(\textit{no goals scored in }]0, t[\,)$ ***R1***

$\Rightarrow F(t) = 1 - \mathrm{e}^{\frac{-t}{36}}$ ***A1***

$f(t) = \dfrac{1}{36}\mathrm{e}^{\frac{-t}{36}}$ ***A1***

So T follows an exponential distribution. ***AG***

[5 marks]

Total [15 marks]

4. (a) **METHOD 1**

Lower 95 % significance level value $0.2297 = \hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{225}}$.

where $\hat{p} = \frac{x}{225}$ ***M1A1***

(**OR** upper 95 % significance level value $0.3481 = \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{225}}$)

$\Rightarrow 0.2297 = \hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{225}}$

$\hat{p} - 0.2297 = \frac{1.96}{15}\sqrt{\hat{p}(1-\hat{p})}$

By GDC $\hat{p} = 0.28892...$ ***A1***

$x = 225 \times 0.2889...$

$x = 65$ ***A1***

[4 marks]

METHOD 2

Interval is symmetric about $\frac{x}{225}$ ***M1A1***

So $\frac{x}{225} = \frac{0.2297 + 0.3481}{2} = 0.2889$ ***A1***

$x = 65$ ***A1***

[4 marks]

(b) p is the probability of getting a head.

$H_0 : p = \frac{1}{2}$; $H_1 : p \neq \frac{1}{2}$ ***M1***

(i) For Amanda, X is the number of heads obtained when the coin is tossed.
$X \sim B(3,\ p)$

$P(\text{Type I error}) = P(X = 0 \text{ or } X = 3)$ ***M1***

$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4} = 0.250$ ***A1***

For Roger Y is the number of heads, $Y \sim B(8,\ p)$

$P(\text{Type I error}) = P(Y \geq 6 \text{ or } Y \leq 2)$ ***M1***

$= \binom{8}{6}\left(\frac{1}{2}\right)^8 + \binom{8}{7}\left(\frac{1}{2}\right)^8 + \binom{8}{8}\left(\frac{1}{2}\right)^8 + \binom{8}{2}\left(\frac{1}{2}\right)^8 + \binom{8}{1}\left(\frac{1}{2}\right)^8 + \binom{8}{0}\left(\frac{1}{2}\right)^8$ ***A1***

$= 0.2890625 = 0.289 \quad \left(\frac{37}{128}\right)$ ***A1***

So Amanda's method has the smaller probability of making a Type I error. ***A1***

(ii) $P(\text{Type II error}) = P(3 \leq Y \leq 5)$ when $p = 0.6$ ***M1***

$= \binom{8}{3}(0.6)^3(0.4)^5 + \binom{8}{4}(0.6)^4(0.4)^4 + \binom{8}{5}(0.6)^5(0.4)^3$ ***(A1)***

$= 0.63479808 = 0.635$ ***A1***

[10 marks]

Total [14 marks]

SECTION B

Sets, relations and groups

1. (a) (i) $S_1 = \ x \in \mathbb{Z}^+ \ \big| \, 1 \text{ divides } x$

$= \ 1, 2, 3, \ldots \ = \mathbb{Z}^+$ ***A1***

(ii) $S_2 = \ x \in \mathbb{Z}^+ \ \big| \, 2 \text{ divides } x$

$= \ 2, 4, 6, \ldots$

hence $S_2{}' = \ 1, 3, 5, \ldots$ ***A1***

(iii) $S_3 = \ x \in \mathbb{Z}^+ \ \big| \, 3 \text{ divides } x$

$= \ 3, 6, 9, \ldots$ ***A1***

hence $S_2 \cap S_3 = \ 6, 12, 18, \ldots$ ***A1***

Note: Accept set descriptions such as 'positive multiples of 6'.

(iv) $S_6 = \ x \in \mathbb{Z}^+ \ \big| \, 6 \text{ divides } x$

$= \ 6, 12, 18, \ldots$ ***A1***

hence $\mathrm{S}_6 \setminus S_3 = S_6 \cap S_3{}' = \varnothing$ ***M1A1***

[7 marks]

(b) $(A \setminus B) \cup (B \setminus A) = (A \cap B') \cup (B \cap A')$ ***M1***

$= \ A \cup (B \cap A') \ \cap \ B' \cup (B \cap A')$ ***A1***

$= \ (A \cup B) \cap (A \cup A') \ \cap \ (B' \cup B) \cap (B' \cup A')$ ***A1***

$= (A \cup B) \cap (B' \cup A')$

$= (A \cup B) \cap (B \cap A)'$

$= (A \cup B) \cap (A \cap B)'$ ***A1***

$= (A \cup B) \setminus (A \cap B)$ ***AG***

Note: It is possible to start from the right-hand side.

[4 marks]

Total [11 marks]

2. (a) ${}_1M_2$ but not ${}_2M_1$ so M is not symmetric
and is not therefore an equivalence relation. ***A1R1***

[2 marks]

(b) (i) ${}_xN_x$ as $x^2-2x=x^2-2x$ so reflexive. ***A1***

${}_xN_x$ as $x^2-2x=y^2-2y$ then $y^2-2y=x^2-2x$ so symmetric ***A1***

${}_xN_y \Rightarrow x^2-2x=y^2-2y$ ***A1***

${}_yN_z \Rightarrow y^2-2y=z^2-2z$ ***A1***

$\Rightarrow x^2-2x=z^2-2z$

Hence N is transitive and is therefore an equivalence relation. ***R1***

(ii) Suppose that $y \neq x$ is in the same equivalence class as x, then

$x^2-2x=y^2-2y$ ***M1A1***

$x^2-y^2=2x-2y$

$(x-y)(x+y)=2(x-y)$ ***A1***

$x+y=2,\ x \neq y$

(The equivalence classes are number pairs that add to two). ***A1***

(iii) If $x=1$ the class is 1 . ***A1***

[10 marks]

Total [12 marks]

3. (a) On $]-\infty, 1], |x-2| = -(x-2)$ ***M1***

$\Rightarrow f(x) = x^2 + x - 2$

$f'(x) = 2x + 1$

$= \left(x + \frac{1}{2}\right)^2 - \frac{9}{4}$ ***A1***

Range is $\left[-\frac{9}{4}, \infty\right[$ ***A1***

Since $f(-2) = f(1) = 0$, f is not an injection. ***R1A1***

[5 marks]

(b) $\ln[2, \infty[\,, |x-2| = x-2$,

so $g(x) = x^2 - x + 2$, $g'(x) = 2x - 1$ and $g'(x) > 0$ ***A1***

$\ln[1, 2], x - 2 = 2 - x$,

so $g(x) = x^2 + x - 2$, $g'(x) = 2x + 1$ and $g'(x) > 0$ ***A1***

So g is bijective and has an inverse g^{-1}. ***R1AG***

$\ln\,]1, 2[\,, y = x^2 + x - 2 = \left(x + \frac{1}{2}\right)^2 - \frac{9}{4}$ ***M1A1***

$x = \pm\sqrt{y + \frac{9}{4}} - \frac{1}{2}$ ***A1***

Hence $g^{-1}(x) = +\sqrt{x + \frac{9}{4}} - \frac{1}{2}$ on $[0, 4]$ ***A1***

$\ln\,]2, \infty[\,, y = x^2 - x + 2 = \left(x - \frac{1}{2}\right)^2 + \frac{9}{4}$ ***A1***

$x = \pm\sqrt{y - \frac{9}{4}} + \frac{1}{2}$ ***A1***

Hence $g^{-1}(x) = +\sqrt{x + \frac{9}{4}} - \frac{1}{2}$ on $[0, 4]$ ***A1***

[10 marks]

Total [15 marks]

4. (a)

$*$	P	Q	R	T
P	P	Q	R	T
Q	Q	R	T	P
R	R	T	P	Q
T	T	P	Q	R

A3

Note: Award ***A2*** if one is wrong, ***A1*** if two are wrong, ***A0*** if three or more are wrong.

The table is closed. ***A1***
Identity is P ***A1***
Associativity follows from associativity of composition of permutations. ***A1***
Inverse of P is P, of Q is T, of R is R and of T is Q ***A1***

$T^1 = T\,;\; T^2 = R\,;\; T^3 = Q\,;\; T^4 = P$ ***A2***

Hence $(S, *)$ is a cyclic group. ***AG***

Note: ***Q*** is also a generator.

[9 marks]

continued …

Question 4 continued

(b) (i) $x \otimes y = x + y + sxy$

$y \otimes x = y + x + syx$ ***M1A1***

Hence $\otimes$ is commutative on $\mathbb{R}$

$(x \otimes y) \otimes z = (x + y + sxy) \otimes z$

$= x + y + sxy + z + sxy + sxz + syz + s^2 xyz$

Since this is symmetrical in x, y, z then $\otimes$ is associative on $\mathbb{R}$. ***M1A1***

(ii) If e is the identity element

$x \otimes e = x + e + sxe = x$ ***M1***

$e(1 + sx) = 0$

$e = 0$ ***A1***

$x \otimes x^{-1} = x + x^{-1} + sxx^{-1} = 0$ ***M1***

$x^{-1}(1 + sx) = -x$

$x^{-1} = \dfrac{-x}{1 + sx}$ ***A1***

There is no inverse for $x = -\dfrac{1}{s} = t$ ***A1***

(iii) Yes since $\mathbb{R} \setminus t, \otimes$ is closed,

associative, has an identity element,

each element has a unique inverse and

$\otimes$ is commutative. ***A1R1***

Using

$x \otimes x = 2x + sx^2 = 0$ ***M1***

$x = \left\{0, -\dfrac{2}{s}\right\}$ ***A1***

[13 marks]

Total [22 marks]

SECTION C

Series and differential equations

1 (a) (i) Domain $-1, 1$ ***A1***

Range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ***A1***

(ii) $f(x) = \arcsin x, \quad f(0) = 0$ ***A1***

$f'(x) = \frac{1}{\sqrt{1-x^2}}, \quad f'(0) = 1$ ***A1***

$f''(x) = -\frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x), \quad f''(0) = 0$ ***A1***

$f'''(x) = -\frac{3}{2}x(1-x^2)^{-\frac{5}{2}}(-2x) + (1-x^2)^{-\frac{3}{2}},$ ***A1***

$f'''(0) = 1$ ***A1***

$\Rightarrow f(x) = x + \frac{x^3}{6} + \ldots$ ***A1***

[8 marks]

(b) $\cos(\arcsin x) = 1 - \frac{\left(x + \frac{x^3}{6}\right)^2}{2} + \frac{\left(x + \frac{x^3}{6}\right)^4}{24}$ ***M1A1***

$= 1 - \frac{\left(x^2 + \frac{x^4}{3} + \ldots\right)}{2} + \frac{x^4 + \ldots}{24}$ ***A1***

$= 1 - \frac{x^2}{2} - \frac{x^4}{8}$ ***A1***

[4 marks]

(c) (i) $p^r(1 + \frac{q}{p}x^2)^r = p^r\left(1 + r\frac{q}{p}x^2 + \frac{r(r-1)}{2}\frac{q^2}{p^2}x^4\right)$ ***M1A1***

(ii) Equating: $p^r = 1 \Rightarrow p = 1;\ rq = -\frac{1}{2} \Rightarrow q = -\frac{1}{2r};\ \frac{r(r-1)}{2}q^2 = -\frac{1}{8}$ ***M1***

$\frac{r(r-1)}{2} \times \frac{1}{4r^2} = -\frac{1}{8} \Rightarrow \frac{r-1}{r} = -1$ ***M1***

$r - 1 = -r$

$r = \frac{1}{2};\ q = -1$ ***A1***

Series is $(1-x^2)^{1/2}$

The same function is being considered in (b) and (c) **R1**

since $\cos(\arcsin x) = \cos\ \arccos\sqrt{(1-x^2)}\ = (1-x^2)^{1/2}$ **R1** ***[7 marks]***

Total [19 marks]

2. (a) $\lim_{x\to 0}\left(\frac{\ln(a^2+x^2)}{\ln(a-x^3)}\right)=\frac{\ln a^2}{\ln a}$ ***M1A1***

$=\frac{2\ln a}{\ln a}=2$ ***A1***

[3 marks]

(b) $\lim_{x\to 0}\left(\frac{\ln(1+x^2)}{\ln(1-x^2)}\right)=\lim_{x\to 0}\left(\frac{\frac{2x}{1+x^2}}{\frac{-2x}{1-x^2}}\right)$ ***M1A1***

$=\lim_{x\to 0}\left(\frac{-(1-x^2)}{(1+x^2)}\right)=-1$ ***A1***

[3 marks]

(c) $\lim_{x\to 0}\left(\frac{2+x^2-2\cos x}{\mathrm{e}^x+\mathrm{e}^{-x}-2\cos x}\right)=\lim_{x\to 0}\left(\frac{2x+2\sin x}{\mathrm{e}^x-\mathrm{e}^{-x}+2\sin x}\right)$ ***M1A1***

$=\lim_{x\to 0}\left(\frac{2+2\cos x}{\mathrm{e}^x+\mathrm{e}^{-x}+2\cos x}\right)$ ***A1***

$=\frac{4}{4}$ ***A1***

$=1$ ***AG***

Note: The expression "$=\frac{4}{4}$" must be shown to obtain the ***A1***.

[4 marks]

Total [10 marks]

3. (a) $y = vx$

$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x} = F(v)$ ***M1***

$x\frac{\mathrm{d}v}{\mathrm{d}x} = F(v) - v$ ***A1***

This is separable, *i.e.* $\int \frac{\mathrm{d}v}{F(v)-v} = \int \frac{\mathrm{d}x}{x}$ ***A1***

[3 marks]

(b) $X = x-1,\ Y = y-2$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{(X+1)+3(Y+2)-7}{3(X+1)-(Y+2)-1}$ ***M1A1***

$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{X+3Y}{3X-Y} = \frac{1+3\frac{Y}{X}}{3-\frac{Y}{X}}$ ***A1***

This is a homogeneous differential equation.

Using $Y = vX$

$v + X\frac{\mathrm{d}v}{\mathrm{d}X} = \frac{1+3\frac{Y}{X}}{3-\frac{Y}{X}} = \frac{1+3v}{3-v}$ ***A1***

$X\frac{\mathrm{d}v}{\mathrm{d}X} = \frac{1+3v}{3-v} - v$

$X\frac{\mathrm{d}v}{\mathrm{d}X} = \frac{1+3v-3v+v^2}{3-v} = \frac{1+v^2}{3-v}$ ***M1A1***

$\int\frac{\mathrm{d}X}{X} = \int\frac{3-v}{1+v^2}\mathrm{d}v$ ***A1***

$\ln|X| = 3\arctan v - \frac{1}{2}\ln(1+v^2) + C$ ***A1A1***

Note: Award ***A1*** for $3\arctan v$ and ***A1*** for $-\frac{1}{2}\ln(1+v^2)$.

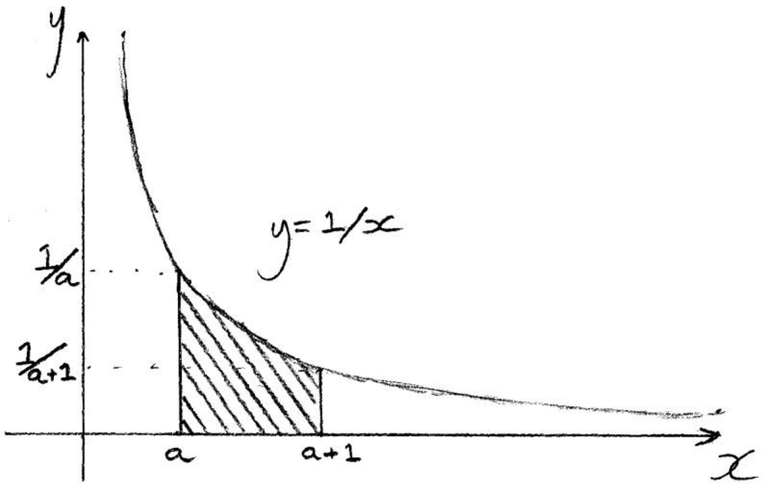
$\ln|x-1| = 3\arctan\left(\frac{y-2}{x-1}\right) - \frac{1}{2}\ln\left(1+\left(\frac{y-2}{x-1}\right)^2\right) + C$ ***A1A1***

Note: Award ***A1*** for each correct substitution.

[11 marks]

Total [14 marks]

4. (a) (i)



(A1) for graph

From consideration of relative areas of rectangle and trapezoid,

$$\frac{1}{(a+1)} < \int_a^{a+1} \frac{dx}{x} < \frac{1}{2}\left(\frac{1}{a} + \frac{1}{a+1}\right)$$ ***M1A1***

$$\int_a^{a+1} \frac{dx}{x} = \left[\ln x\right]_a^{a+1} = \ln\left(\frac{a+1}{a}\right)$$ ***A1***

$$\frac{1}{a+1} < \ln\left(\frac{a+1}{a}\right) < \frac{1}{2}\left(\frac{1}{a} + \frac{1}{a+1}\right)$$ ***AG***

(ii) Putting $a = 1$ ***M1***

$$\frac{1}{2} < \ln 2 < \frac{3}{4}$$ ***AG***

(iii) If $\ln 3 = \ln\left(\frac{a+1}{a}\right)$ ***M1***

$3a = a + 1$

$$a = \frac{1}{2}$$ ***A1***

$$p = \frac{2}{3},\ q = \frac{1}{2}\left(2 + \frac{2}{3}\right) = \frac{4}{3}$$ ***A1***

[8 marks]

continued ...

Question 4 continued

(b) From (a)(i)

$$\frac{1}{n} < \ln n - \ln(n-1) < \frac{1}{2}\left(\frac{1}{n-1} + \frac{1}{n}\right)$$ ***M1A1***

$$\frac{1}{n-1} < \ln(n-1) - \ln(n-2) < \frac{1}{2}\left(\frac{1}{n-2} + \frac{1}{n-1}\right)$$ ***A1***

$$\frac{1}{n-2} < \ln(n-2) - \ln(n-3) < \frac{1}{2}\left(\frac{1}{n-3} + \frac{1}{n-2}\right)$$

. . . . . .

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$$\frac{1}{2} < \ln 2 - \ln 1 < \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2}\right)$$ ***A1***

Summing

$$H_n - 1 < \ln n < \frac{1}{2}\left(H_n - \frac{1}{n}\right) + \frac{1}{2}(H_n - 1)$$ ***A1***

$$H_n - 1 < \ln n < H_n - \frac{1}{2} - \frac{1}{2n}$$ ***AG***

[5 marks]

(c) $\gamma_n - \gamma_{n-1} = H_n - \ln n - H_{n-1} + \ln(n-1)$ ***M1A1***

$= H_n - H_{n-1} - \ln\frac{n}{n-1}$ ***A1***

$= \frac{1}{n} - \ln\frac{n}{n-1} < 0$, using the result of (a) (i) ***A1***

Hence the terms decrease as n increases.

[4 marks]

Total [17 marks]

SECTION D

Discrete mathematics

1. (a) (i) **EITHER**

$$\frac{1001}{512} = 1rem489$$
$$\frac{489}{64} = 7rem41$$
$$\frac{41}{8} = 5rem1$$
$$\Rightarrow 1001_{ten} = 1751_8$$ ***M1A1***

OR

$$\begin{array}{r|rr} 8 & \underline{1001} & 1 \\ 8 & \underline{125} & 5 \\ 8 & \underline{15} & 7 \\ & 1 & \end{array}$$

$$\Rightarrow 1001_{ten} = 1751_8$$ ***M1A1***

(ii) Let the octal number be

$$a_n \times 8^n + a_{n-1} \times 8^{n-1} + a_{n-2} \times 8^{n-2} + \ldots + a_0 \times 8^0$$ ***M1A1***

$$= a_n(8^n - 1) + a_{n-1}(8^{n-1} - 1) + \ldots + a_0(8^0 - 1) + \sum_{i=0}^{n} a_i$$ ***M1A1***

where $\sum_{i=0}^{n} a_i$ is the sum of the digits

But

$(8^n - 1) = (7+1)^n - 1 = \text{multiple of } 7$ ***R1***

$\left[\text{OR } (8^n - 1) = (8-1)(8^{n-1} + 8^{n-2} + 8^{n-3} + \ldots + 1) = \text{multiple of } 7\right]$

Hence the octal number is divisible by 7
if and only if the sum of the digits is divisible by 7. ***R1***

(iii) $1001_{ten} = 1571_8$ and $1+5+7+1=14$
Since 14 is divisible by 7 then so is 1001_{ten}. ***R1***

[9 marks]

continued ...

Question 1 continued

(b) Let $m = 9 \times 5 = 45$

$M_1 = \frac{m}{9} = 5$; $M_2 = \frac{m}{5} = 9$ ***M1A1***

Solving $5x_1 \equiv 1 \pmod 9 \Rightarrow x_1 \equiv 2 \pmod 9$ ***A1***

$9x_2 \equiv 1 \pmod 5 \Rightarrow x_2 \equiv 4 \pmod 5$ ***A1***

So $x \equiv 4 \times 5 \times 2 + 3 \times 9 \times 4 \pmod{45}$ ***A1***

$\equiv 148 \pmod{45}$

$\equiv 13 \pmod{45}$ ***A1***

[6 marks]

Total [15 marks]

2. (a)

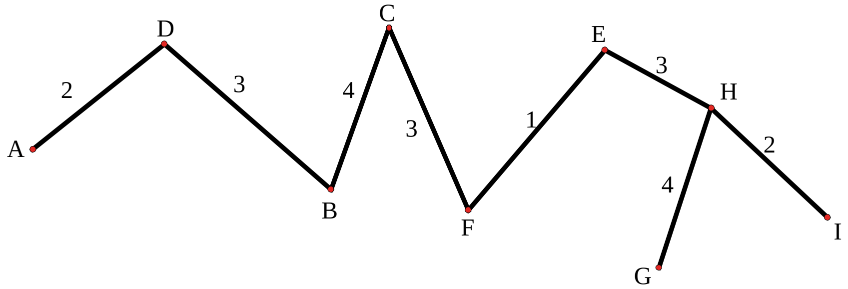
For minimum spanning tree T :
start at A:

delete row A, choose least value in column A	2 in row D:	AD is in T	***M1A1***
delete row D, least value in column D	3 in row B:	DB is in T	
delete row B, least value in column B	4 in row C:	BC is in T	
delete row C, least value in column C	3 in row F:	CF is in T	
delete row F, least value in column F	1 in row E:	FE is in T	
delete row E, least value in column E	3 in row H:	EH is in T	
delete row H, least value in column H	2 in row I:	HI is in T	***A4***
add smallest edge to G	4	HG is in T	***A1***

Note: Award ***A4*** if all other edges are correct, ***A3*** if one wrong, ***A2*** if two wrong, ***A1*** if three wrong, ***A0*** if four wrong.

[7 marks]

(b)



A1

Minimum spanning tree T

Total weight $= 2 + 3 + 4 + 3 + 1 + 3 + 4 + 2 = 22$ ***A1***

[2 marks]

Total [9 marks]

3. (a) Since every edge has two ends it must contribute exactly 2 to the degree sum. ***A1***

Hence the vertex sum of a graph is twice the number of edges. ***AG***

[1 mark]

(b) Let V_1 and V_2 be the sets of even degree vertices and odd degree vertices in G respectively.

Then $2e = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v)$ ***M1A1***

Since the right hand side is even and $\sum_{v \in V_1} \deg(v)$ is even then $\sum_{v \in V_2} \deg(v)$ must be even. ***R1***

But each term of $\sum_{v \in V_2} \deg(v)$ is odd so there must be an even number of such terms, ***R1***

i.e. G must have an even numbers of vertices of odd degree. ***AG***

[4 marks]

continued …

Question 3 continued

(c) (i) For graph G with vertex set V
and n vertices we have (with the usual notation),

$v - e + f = 2$ ***M1***

If $f = 4$

$n - e = -2$ ***A1***

$e - n = 2$

$2e - 2n = 4$ ***A1***

From part (a) $\sum_{v \in V} \deg(v) = 2e$

So $\sum_{v \in V} \deg(v) - 2v = 4$ ***M1***

G has $(n-1)$ vertices degree 3 and one vertex degree d

$3(n-1) + d - 2n = 4$ ***A1***

So $3n - 3 + d - 2n = 4$

$n + d = 7$ ***A1***

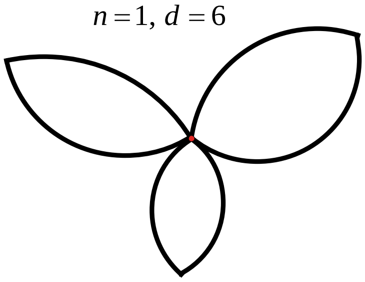
Hence

$(n, d) = (1, 6), (2, 5), (3, 4), (5, 2)$ or $(6, 1)$ ***A1A1A1***

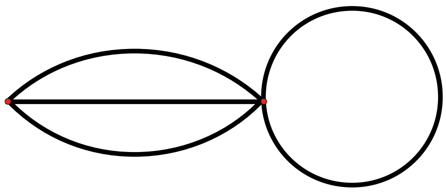
Note: $(n, d) = (4, 3)$ not possible.

(ii)

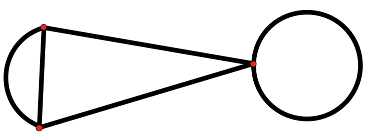
$n = 1, d = 6$



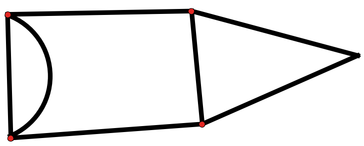
$n = 2, d = 5$



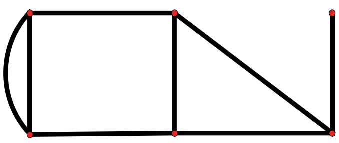
$n = 3, d = 4$



$n = 5, d = 2$



$n = 6, d = 1$



A1A1A1A1A1

[14 marks]

Total [19 marks]

4. (a) Let $f(n) = 10^n + 3 \times 4^{n+2} + 5$

$f(1) = 10 + 192 + 5 = 207 = 23 \times 9$ ***M1A1***

$f(n+1) = 10^{n+1} + 3 \times 4^{n+3} + 5$ ***A1***

$= 10(10^n + 3 \times 4^{n+2} + 5) - 18 \times 4^{n+2} - 45$ ***A1***

$\Rightarrow f(n+1) - 10f(n) = 9(-2 \times 4^{n+2} - 5)$ ***M1A1***

Hence if $f(n)$ is divisible by 9 then so is $f(n+1)$ and since $f(1)$ is divisible by 9 ***R2***
then $f(n)$ is divisible by 9 $\forall n \in \mathbb{Z}^+$. ***AG***

[8 marks]

(b) (i) $ax \equiv b \pmod p$

$a^{p-2}ax \equiv a^{p-2}b \pmod p$ ***M1***

$a^{p-1}x \equiv a^{p-2}b \pmod p$ ***A1***

By Fermat's Little Theorem $a^{p-1} \equiv 1 \pmod p$ ***R1***

Hence

$x \equiv a^{p-2}b \pmod p$ ***AG***

$4x \equiv 3 \pmod 7$

$x \equiv 4^5 \times 3 \pmod 7$ ***A1***

$x \equiv 3072 \pmod 7$

$x \equiv 6 \pmod 7$ ***A1***

(ii) $5^6 \equiv 1 \pmod 9$ ***M1***

$(5^6)^{25} \equiv 1 \pmod 9$ ***A1***

$(5^6)^{25} \times 5^5 \equiv 5^5 \pmod 9$ ***A1***

$5^{155} \pmod 9 \equiv 2 \pmod 9$ ***A1***

so last digit is 2

[9 marks]

Total [17 marks]