IB DIPLOMA PROGRAMME
PROGRAMME DU DIPLÔME DU BI
PROGRAMA DEL DIPLOMA DEL BI

M07/5/MATHL/HP3/ENG/TZ2/XX/M+

MARKSCHEME

May 2007

MATHEMATICS

Higher Level

Paper 3

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**: often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). Apply a **MR** penalty of 1 mark to that question. Award the marks as usual and then write –1(**MR**) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled **(d)** and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$f'(x) = \ 2\cos(5x-3)\ \ 5 \quad = 10\cos(5x-3)$ ***A1***

Award ***A1*** for $2\cos(5x-3)\ \ 5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

*Candidates should be penalized **once only IN THE PAPER** for an accuracy error **(AP)**. Award the marks as usual then write **(AP)** against the answer. On the **front** cover write –1**(AP).** Deduct 1 mark from the total for the paper, not the question*.

- If a final correct answer is incorrectly rounded, apply the ***AP***.
- If the level of accuracy is not specified in the question, apply the ***AP*** for correct answers not given to three significant figures.

If there is no working shown, and answers are given to the correct two significant figures, apply the ***AP***. However, do **not** accept answers to one significant figure without working.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

SECTION A

Statistics and probability

1. (a) $P(X \leq 4) = \frac{1}{4} + \frac{1}{4}\left(1 - \frac{1}{4}\right) + \frac{1}{4}\left(1 - \frac{1}{4}\right)^2 + \frac{1}{4}\left(1 - \frac{1}{4}\right)^3$ ***M1A1***

$= 0.684 \quad \left(\text{accept } \frac{175}{256}\right)$ ***A1*** ***N3***

[3 marks]

(b) This is a negative binomial distribution $\text{NB}\left(8, \frac{1}{4}\right)$

$P(X = 20) = \binom{19}{7}\left(\frac{1}{4}\right)^8\left(\frac{3}{4}\right)^{12}$ ***M1A1***

$= 0.0244$ ***A1*** ***N3***

[3 marks]

(c) **EITHER**

The geometric distribution in (a) is the same as the negative binomial distribution in (b) when X is the number of trials to get one success. ***A2***

OR

$\text{NB}\left(8, \frac{1}{4}\right)$ is the sum of 8 geometric distributions with the same p value of $\frac{1}{4}$. ***A2***

[2 marks]

Total [8 marks]

2. (a) Since the sample is large we can use a normal approximation *(R1)*

$\hat{p} = \frac{580}{1100} = 0.527...$ *(M1)(A1)*

A 95 % confidence interval is

$0.5273 \pm 1.96\sqrt{\frac{0.5273 \times (1 - 0.5273)}{1100}}$ *M1(A1)*

$= 0.5273 \pm 0.0295$

$=$ 0.498, 0.557 Accept 0.497, 0.557 ***A1A1*** ***N4***

[7 marks]

(b) The interval width is $2 \times 1.96\sqrt{\frac{0.5273 \times 0.4727}{n}}$ ***M1A1A1***

$2 \times 1.96\sqrt{\frac{0.5273 \times 0.4727}{n}} < 0.02$ ***M1***

$2 \times 1.96\sqrt{\frac{0.2493}{n}} < 0.02$

so

$\left(\frac{2 \times 1.96\sqrt{0.2493}}{0.02}\right)^2 < n$ ***A1***

$9575.4 < n$ ***A1***

So n must be at least 9576 or 9580 (to 3 s.f.) ***R1*** ***N3***

[7 marks]

Total [14 marks]

3. Finding the differences ***M1***

Competitor	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Time before training	80	62	45	73	65	53	61	48	81	50	50	29	52	33	71
Time after training	85	74	60	67	69	55	68	46	89	60	64	26	61	33	72
Difference d	5	12	15	–6	4	2	7	–2	8	10	14	–3	9	0	1

A2

H_0 : the training schedule does not help improve times. Or $\mu = 0$. Accept $\mu_1 = \mu_2$. ***A1***

H_1 : the training schedule does help improve times. Or $\mu > 0$. Accept $\mu_2 > \mu_1$. ***A1***

EITHER

$n = 15,\ \sum d = 76,\ \sum d^2 = 954$ **OR** $\bar{d} = \frac{76}{15} = 5.07$ ***(A1)***

$s_{n-1}^2 = \frac{1}{14}\left(954 - \frac{76^2}{15}\right) = 40.638$ **OR** $s_{n-1} = 6.37$ ***(M1)(A1)***

(Small sample) so use a one-sided t-test. ***(A1)***

$t = \dfrac{\frac{76}{15}}{\sqrt{\frac{40.638}{15}}} = 3.078$ ***M1A1*** ***N6***

$\nu = 14$, ***A1***

At the 1 % level the critical value is 2.624. ***A1***

Since $2.624 < 3.078$ H_0 is rejected and there is evidence to support the claim. ***R1*** ***N0***

OR

$p = 0.00409$ ***A8***

Since $0.00409 < 0.01$ H_0 is rejected and there is evidence to support the claim. ***R1*** ***N0***

[14 marks]

4. (a) $\mathrm{P}(Y=y)=\mathrm{e}^{-\mu}\dfrac{\mu^{y}}{y!}$

$\mathrm{P}(Y=y+1)=\mathrm{e}^{-\mu}\dfrac{\mu^{y+1}}{(y+1)!}$ ***(A1)***

$\dfrac{\mathrm{P}(Y=y+1)}{\mathrm{P}(Y=y)}=\dfrac{\mathrm{e}^{-\mu}\dfrac{\mu^{y+1}}{(y+1)!}}{\mathrm{e}^{-\mu}\dfrac{\mu^{y}}{y!}}=\dfrac{\mathrm{e}^{-\mu}\mu^{y+1}y!}{\mathrm{e}^{-\mu}\mu^{y}(y+1)!}$ ***M1***

$=\dfrac{\mu}{(y+1)}$ ***A1***

$\mathrm{P}(Y=y+1)=\dfrac{\mu}{(y+1)}\mathrm{P}(Y=y)$ ***AG***

[3 marks]

(b) H_0 : The data can be modeled by a Poisson distribution.
H_1 : The data cannot be modeled by a Poisson distribution.

$\sum f=80,\ \dfrac{\sum fx}{\sum f}=\dfrac{0\times 4+1\times 18+2\times 19+\ldots+5\times 8}{80}=\dfrac{200}{80}=2.5$ ***A1***

Theoretical frequencies are

$f(0)=80.\mathrm{e}^{-2.5}=6.5668$ ***(M1)A1***

$f(1)=\dfrac{2.5}{1}\times 6.5668=16.4170$ ***A1***

$f(2)=\dfrac{2.5}{2}\times 16.4170=20.5212$

$f(3)=\dfrac{2.5}{3}\times 20.5212=17.1010$

$f(4)=\dfrac{2.5}{4}\times 17.1010=10.6882$ ***A1***

Note: Award ***A1*** for $f(2),\ f(3),\ f(4)$.

$f(5\text{ or more})=80-(6.5668+16.4170+20.5212+17.1010+10.6882)$ ***A1***
$=8.7058$

Number of cars	0	1	2	3	4	5 or more
O	4	18	19	20	11	8
E	6.5668	16.4170	20.5212	17.1010	10.6882	8.7058

$$\chi^2=\frac{(4-6.5668)^2}{6.5668}+\frac{(18-16.4170)^2}{16.4170}+\frac{(19-20.5212)^2}{20.5212}+\frac{(20-17.1010)^2}{17.1010}+\frac{(11-10.6882)^2}{10.6882}+\frac{(8-8.7058)^2}{8.7058}$$

$=1.83$ (accept 1.82) ***(M1)A1***

$\nu=4$ (six frequencies and two restrictions) ***(A1)***

$\chi^2(4)=9.488$ at the 5 % level. ***A1***

Since $1.83<9.488$ we accept H_0 and conclude that the distribution can be modeled by a Poisson distribution. ***R1*** ***N0***

[11 marks]

Total [14 marks]

5. (a) $\mathrm{E}(U-3V)=\mathrm{E}(U)-3\mathrm{E}(V)$

$=66-57=9$ ***A1***

$\mathrm{Var}(U-3V)=\mathrm{Var}(U)+9\mathrm{Var}(V)$ ***M1***

$=5+27=32$ ***A1***

$\mathrm{P}(U>3V)=\mathrm{P}(U-3V>0)$ ***M1***

$=\mathrm{P}\left(Z>\frac{0-9}{\sqrt{32}}\right)$ ***(M1)***

$=\mathrm{P}(Z>-1.5909...)$

$=1-\varphi(-1.5909...)$

$=0.944$ ***A1*** ***N0***

[6 marks]

(b) $\mathrm{E}\left[\left(X-\mathrm{E}(X)\right)^2\right]=\mathrm{E}\left[X^2-2X\,\mathrm{E}(X)+\left(\mathrm{E}(X)\right)^2\right]$ ***M1***

$=\mathrm{E}(X^2)-2\mathrm{E}(X)\mathrm{E}(X)+\left(\mathrm{E}(X)\right)^2$ ***A1***

$=\mathrm{E}(X^2)-\left(\mathrm{E}(X)\right)^2$ ***A1***

Since $\mathrm{E}\left[\left(X-\mathrm{E}(X)\right)^2\right]\geq 0$ ***M1***

$\mathrm{E}(X^2)\geq\left(\mathrm{E}(X)\right)^2$ ***AG***

[4 marks]

Total [10 marks]

SECTION B

Sets, relations and groups

1. Consider, for example ***M1***
$f(1)=3$
$f(4)=3$ ***A1***
Since $f(1)=f(4)$, f is not an injection. ***R1***

Let $a \in \mathbb{Z}$, and suppose that there is an even x such that $f(x)=a$. ***M1***
Then $\frac{x+2}{2}=a$ ***A1***
$x=2a-2$ ***A1***
This x is even (and $x \in \mathbb{Z}$) and $f(x)=a$ ***R1***
thus f is a surjection. ***AG***

[7 marks]

2. Reflexive: If $(x, y) \in \mathbb{R}^2$ then $x^2-y^2=x^2-y^2$ so $(x, y)R(x, y), \forall (x, y) \in \mathbb{R}^2$ ***A1***
Symmetric:
If $(x, y)R(p, q)$ then $x^2-y^2=p^2-q^2$, ***M1***
so $p^2-q^2=x^2-y^2$ and $(p, q)R(x, y)$ ***A1***
Transitive:
If $(x, y)R(p, q)$ and $(p, q)R(v, w)$,
then $x^2-y^2=p^2-q^2$ and $p^2-q^2=v^2-w^2$ so $x^2-y^2=v^2-w^2$. ***M1***
Hence $(x, y)R(v, w)$ ***A1***
$\overline{(1, 1)} = (x, y)\,|\,(x, y)R(1, 1)$
$= (x, y)\,|\,x^2-y^2=1^2-1^2=0$ ***M1A1***
$= (x, y)\,|\,y=\pm x$ ***A1***
Thus the equivalence class of $(1, 1)$ is a pair of straight lines through the origin, ***A1***
with slopes ± 1 (or perpendicular). ***A1***

[10 marks]

3. (a) If $P(n):(bab^{-1})^n = ba^n b^{-1}$

for $n=1$, $P(1): bab^{-1} = bab^{-1}$ so $P(1)$ is true ***A1***

assume $P(k)$ is true, *i.e.* $(bab^{-1})^k = ba^k b^{-1}$ ***M1***

for $n = k+1$,

$(bab^{-1})^{k+1} = (bab^{-1})^k (bab^{-1})$ ***M1A1***

$= ba^k b^{-1} bab^{-1}$ ***A1***

$= ba^k eab^{-1} = ba^k ab^{-1}$ ***A1***

$= ba^{k+1} b^{-1}$ ***A1***

Hence $P(k) \Rightarrow P(k+1)$ and $P(1)$ is true, so $P(n)$ is true for all $n \in \mathbb{Z}^+$. ***R1***

[8 marks]

(b) **EITHER**

$(bab^{-1})(ba^{-1}b^{-1})$ ***M1***

$= ba(b^{-1}b)a^{-1}b^{-1}$ using associativity

$= baea^{-1}b^{-1}$

$= baa^{-1}b^{-1} = e$ ***A1***

Therefore by definition of inverse $(bab^{-1})^{-1} = ba^{-1}b^{-1}$ ***R1***

[3 marks]

OR

Using the reversal rule ***(M1)***

$(bab^{-1})^{-1} = (b^{-1})^{-1} a^{-1} b^{-1}$ ***A2***

$= ba^{-1}b^{-1}$ ***AG***

[3 marks]

(c) Let $n = -m$, where $m \in \mathbb{Z}^+$ ***M1***

Then $(bab^{-1})^n = (bab^{-1})^{-m}$

$= \left((bab^{-1})^{-1}\right)^m$ ***M1***

$= (ba^{-1}b^{-1})^m$ using part (b) ***A1***

$= (ba^{-m}b^{-1})$ using part (a) ***A1***

$= ba^n b^{-1}$ ***AG***

[4 marks]

Total [15 marks]

4. (a) Matrices are associative under multiplication. *A1*

Using $a=1, b=0$ we have $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as the identity element (for ***M***). *A1*

$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} = \frac{1}{a^2+b^2}\begin{pmatrix} a & -(-b) \\ -b & a \end{pmatrix}$ so each element has an inverse (belonging to ***M***) since $a^2+b^2 \neq 0$ *M1A1*

$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} s & -t \\ t & s \end{pmatrix} = \begin{pmatrix} as-bt & -(at+bs) \\ bs+at & as-bt \end{pmatrix} = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ *A1*

and $\begin{vmatrix} p & -q \\ q & p \end{vmatrix} = (p^2+q^2) = (a^2+b^2)\times(s^2+t^2) \neq 0$ showing closure. *A1*

Hence $\{M, \times\}$ forms a group. *AG*

[6 marks]

(b) Let G be the multiplicative group of non-zero complex numbers

Define $\phi : G \to M$ by $\phi(a+\mathrm{i}b) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ where $a+\mathrm{i}b \in G$ *M1A1*

ϕ is a bijection *R1*

Let $x = a+\mathrm{i}b$ and $y = c+\mathrm{i}d$ then

$\phi(xy) = \phi\ (a+\mathrm{i}b)(c+\mathrm{i}d)$ *M1*

$= \phi\ (ac-bd)+\mathrm{i}(bc+bd)$ *A1*

$= \begin{pmatrix} ac-bd & -bc-bd \\ bc+bd & ac-bd \end{pmatrix}$ *A1*

$= \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$ *A1*

$= \phi(x)\ \phi(y)$ *A1*

and hence is an isomorphism. *AG*

[8 marks]

Total [14 marks]

5. **EITHER**

Suppose the group is $G, *$

If g is a generator then so is g^{-1} (since if $x \in G$ then $x = g^n = (g^{-1})^{-n}$ for $n \in \mathbb{Z}$) *M1A1*

If $G, *$ has only one generator then $g = g^{-1}$ and *M1*

$g * g = g * g^{-1} = e$ *A1*

Hence G is of order two which is less than three. *R1*

This contradiction shows that G must have more than one generator. *R1*

OR

Consider a cyclic group of order n. *M1*

If g is a generator then so is g^{-1} (since if $x \in G$ then $x = g^n = (g^{-1})^{-n}$ for $n \in \mathbb{Z}$) *M1A1*

Since $n \geq 3$, g is not equal to g^{-1}. *M1A1*

Therefore there are at least two generators, g and g^{-1}. *R1*

[6 marks]

6. If $x \in A \times (B \cup C)$, then $x = (a,\ p)$ where $a \in A$ and $p \in B$ or C ***M1***

i.e. $x \in A \times B$ or $x \in A \times C$ ***A1***

$\therefore\quad A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ ***A1***

If $x \in (A \times B) \cup (A \times C)$, then $x = (a, b)$ or $x = (a, c)$ ***M1***

i.e. *x* has the form (element of *A*, element of $(B \cup C)$) ***A1***

i.e. $x \in A \times (B \cup C)$

$\therefore\quad (A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$ ***A1***

Since $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ and $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

then $A \times (B \cup C) = (A \times B) \cup (A \times C)$ ***R2***

[8 marks]

SECTION C

Series and differential equations

1. (a) $\lim_{s\to 4}\left[\dfrac{s-\sqrt{3s+4}}{(4-s)}\right]=$

$$=\lim_{s\to 4}\left[\frac{1-\frac{3}{2}(3s+4)^{-\frac{1}{2}}}{-1}\right] \text{ using l'Hôpital's Rule} \qquad \textit{M1A1 A1}$$

$$=\left[\frac{1-\frac{3}{2}(16)^{-\frac{1}{2}}}{-1}\right]$$

$$=\left[\frac{1-\frac{3}{8}}{-1}\right]$$

$$=-\frac{5}{8} \qquad \textit{A1} \qquad \textit{N1}$$

[4 marks]

(b) **EITHER**

$$\lim_{x\to 0}\left(\frac{\cos x-1}{x\sin x}\right)$$

$$=\lim_{x\to 0}\left(\frac{-\sin x}{x\cos x+\sin x}\right) \qquad \textit{M1A1}$$

$$=\lim_{x\to 0}\left(\frac{-\cos x}{\cos x-x\sin x+\cos x}\right) \qquad \textit{M1A1}$$

$$=-\frac{1}{2} \qquad \textit{A1} \qquad \textit{N1}$$

[5 marks]

OR

$$\lim_{x\to 0}\left(\frac{\cos x-1}{x\sin x}\right)=\lim_{x\to 0}\left(\frac{1-2\sin^2\frac{x}{2}-1}{2x\sin\frac{x}{2}\cos\frac{x}{2}}\right) \qquad \textit{M1A1}$$

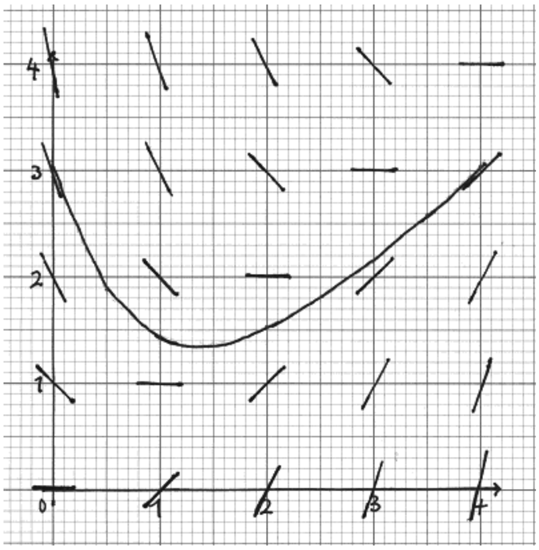
$$=\lim_{x\to 0}\left(\frac{-\sin\frac{x}{2}}{x\cos\frac{x}{2}}\right)=\lim_{x\to 0}\left(\frac{-\tan\frac{x}{2}}{x}\right) \qquad \textit{M1A1}$$

$$=\lim_{x\to 0}\left(\frac{-\frac{1}{2}\sec^2\frac{x}{2}}{1}\right)=-\frac{1}{2} \qquad \textit{A1} \qquad \textit{N1}$$

[5 marks]

Total [9 marks]

2. (a)



A1A1A1

Note: Award ***A1*** for attempt of slope field, ***A1*** for parallel line segments at appropriate points, ***A1*** for completely correct.

[3 marks]

(b) For curve (candidate's minimum should be approximately on the line $y = x$). ***A1***

[1 mark]

(c) $\frac{\mathrm{d}y}{\mathrm{d}x} = x - y$

$\frac{\mathrm{d}y}{\mathrm{d}x} + y = x$ ***(A1)***

integrating factor is $\mathrm{e}^{\int \mathrm{d}x} = \mathrm{e}^x$ ***M1A1***

so

$\mathrm{e}^x \frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^x y = \mathrm{e}^x x$ ***(A1)***

$\mathrm{e}^x y = \int x\mathrm{e}^x \,\mathrm{d}x$ ***A1***

$= x\mathrm{e}^x - \int \mathrm{e}^x \,\mathrm{d}x$ ***M1A1***

$= x\mathrm{e}^x - \mathrm{e}^x + c$ ***A1***

$(0, 3)$ belongs to the curve so

$3 = -1 + c$

$c = 4$ ***A1***

and $y = x - 1 + 4\mathrm{e}^{-x}$ ***A1***

[10 marks]

Total [14 marks]

3. (a) Let $\frac{1}{16r^2+8r-3}=\frac{A}{4r-1}+\frac{B}{4r+3}$

substituting values of r ***(M1)***

$A=\frac{1}{4}$ $\quad B=-\frac{1}{4}$ ***M1A1A1*** ***N4***

[4 marks]

(b) $\frac{1}{16r^2+8r-3}=\frac{1}{4}\left(\frac{1}{4r-1}-\frac{1}{4r+3}\right)$

substituting values of $r=1, 2, 3, ..., n$ gives

$S_n=\frac{1}{4}\left(\frac{1}{3}-\frac{1}{7}+\frac{1}{7}-\frac{1}{11}+\frac{1}{11}-\frac{1}{15}+...+\frac{1}{4n-1}-\frac{1}{4n+3}\right)$ ***M1A1A1***

$=\frac{1}{4}\left(\frac{1}{3}-\frac{1}{4n+3}\right)$ ***A1***

[4 marks]

(c) $\sum_{r=1}^{\infty}\left(\frac{1}{16r^2+8r-3}\right)=\lim_{n=\infty}\frac{1}{4}\left(\frac{1}{3}-\frac{1}{4n+3}\right)$ ***M1***

$=\frac{1}{12}$ ***A1***

Hence the series is convergent. ***AG***

[2 marks]

Total [10 marks]

4. $\int_1^{\infty}\frac{1}{x^p}\mathrm{d}x=\lim_{a\to\infty}\int_1^a\frac{1}{x^p}\mathrm{d}x$ ***M1***

$=\lim_{a\to\infty}\left[\frac{1}{1-p}x^{(1-p)}\right]_1^a, \ (p\neq 1)$ ***A1***

$=\lim_{a\to\infty}\left[\frac{1}{1-p}\ a^{(1-p)}-1\right]$ ***A1***

if $p<1,\ 1-p>0$ and $\left[\frac{1}{1-p}\ a^{(1-p)}-1\right]\to\infty$ ***R1***

if $p>1,\ a^{(1-p)}\to 0$ as $a\to\infty$ so $\lim_{a\to\infty}\left[\frac{1}{1-p}\ a^{1-p}\ -1\right]\to\frac{1}{p-1}$ ***A1***

if $p=1,\ \int_1^{\infty}\frac{1}{x}\mathrm{d}x=\ \ln x\ _1^{\infty}\to\infty$ ***A1***

Hence $\int_1^{\infty}\frac{1}{x^p}\mathrm{d}x$ converges for $p>1$. ***A1***

[7 marks]

5. (a) (i) $\frac{dy}{dx} = (1-2x)y$

$\frac{d^2y}{dx^2} = (1-2x)\frac{dy}{dx} - 2y$ ***M1A1***

$\frac{d^3y}{dx^3} = (1-2x)\frac{d^2y}{dx^2} - 4\frac{dy}{dx}$ ***A1***

$\frac{d^4y}{dx^4} = (1-2x)\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 4\frac{d^2y}{dx^2}$ ***A1***

$= (1-2x)\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2}$ ***AG***

(ii) $y(0) = 2$

$y^1(0) = 2$ ***A1***

$y^2(0) = 2 - 4 = -2$ ***A1***

$y^3(0) = -2 - 8 = -10$ ***A1***

$y^4(0) = -10 + 12 = 2$ ***A1***

$y = 2 + \frac{2x}{1!} - \frac{2x^2}{2!} - \frac{10x^3}{3!} + \frac{2x^4}{4!} + \ldots$ ***M1A1***

Note: Award ***M1*** if at least three terms are given.

[10 marks]

(b) $y(0.5) = 2 + 1 - 0.25 - 0.208333 + 0.005208 = 2.55$ ***(M1)A1***

[2 marks]

(c) $\frac{dy}{dx} = (1-2x)y,\ h = 0.1$

$y_{n+1} = y_n + hf(x_n, y_n)$

x_i	y_i	$\frac{dy}{dx}$	δy	
0	2	2	0.2	***M1***
0.1	2.2	1.76	0.176	***A1***
0.2	2.376	1.4256	(0.14256)	***A1***
0.3	2.51856	1.007424	(0.1007424)	***A1***
0.4	2.6193024	0.52386048	(0.052386048)	***A1***
0.5	2.6716884			

$y(0.5) = 2.67$ ***A1*** ***N0***

[6 marks]

(d) The Maclaurin method can be made more accurate by taking more terms. ***A1***
Euler's method can be made more accurate by decreasing the step value. ***A1***

[2 marks]

Total [20 marks]

SECTION D

Discrete mathematics

1. **METHOD 1**

$(x+y)^p = x^p + \binom{p}{1}x^{p-1}y + \binom{p}{2}x^{p-2}y^2 + \binom{p}{3}x^{p-3}y^3 + \ldots + y^p$ ***M1A1***

Each of the coefficients $\binom{p}{1}, \binom{p}{2}, \binom{p}{3}, \ldots$ is an integer ***R1***

and is a multiple of p ***R1***

because p is a prime ***R1***

$(x+y)^p = x^p + y^p + \text{a multiple of } p$ ***R1***

$(x+y)^p \equiv (x^p + y^p)(\text{mod } p)$ ***AG***

[6 marks]

METHOD 2

By Fermat's little theorem: $(x+y)^p \equiv (x+y)(\text{mod } p)$ ***M1A1***

$x^p \equiv x(\text{mod } p)$, $y^p \equiv y(\text{mod } p)$ ***A1A1***

$(x+y)^p \equiv (x+y)(\text{mod } p)$

$\equiv x(\text{mod } p) + y(\text{mod } p)$ ***M1***

$\equiv x^p(\text{mod } p) + y^p(\text{mod } p)$ ***A1***

$\equiv (x^p + y^p)(\text{mod } p)$ ***AG***

[6 marks]

2. (a) $858 = 1 \times 714 + 144$ ***M1A1***

$714 = 4 \times 144 + 138$ ***A1***

$144 = 1 \times 138 + 6$ ***A1***

$138 = 23 \times 6 + 0$ ***A1***

so $\gcd(858, 714) = 6$ ***A1***

$6 = 144 - 1 \times 138$ ***M1A1***

$= 144 - (714 - 4 \times 144)$ ***M1A1***

$= 5 \times 144 - 714$

$= 5(858 - 1 \times 714) - 714$ ***(A1)***

$= 5 \times 858 - 6 \times 714$ ***A1***

[12 marks]

(b) $\gcd(5, 8) = 1$ so $1 = 2 \times 8 - 3 \times 5$ ***(M1)(A1)***

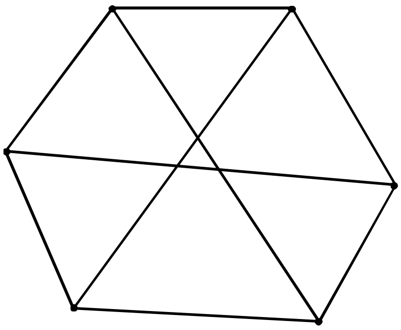
so one solution *i.e.* lattice point is $(-3, 2)$ ***A1***

A complete solution is $x = -3 + 8n$, $y = 2 - 5n$, $(n \in \mathbb{Z})$. ***(M1)A1***

[5 marks]

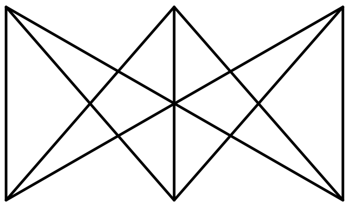
Total [17 marks]

3. (a)



A3

or equivalent
e.g.



[3 marks]

(b) For $K_{3,\,3}$, $v = 6$ and $e = 9$ ***A1***

Suppose a simple planar form exists for $K_{3,\,3}$. ***M1***

$K_{3,\,3}$ has no triangles. ***R1***

$\therefore e \le 2v - 4$ ***A1***

This relation does not hold for $K_{3,3}$ because $9 \not\le 2(6) - 4$ ***R1***

This contradiction shows that $K_{3,3}$ is not planar. ***R1***

[6 marks]

(c) For component *i*, $v_i - e_i + f_i = 2$ ***M1***

Adding for *x* components gives $\sum v_i - \sum e_i + \sum f_i = 2x$ ***M1***

$\sum v_i = v$, $\sum e_i = v$, ***A1***

and total number of faces is the sum of the faces of the individual components minus $(x-1)$, the number of faces counted twice. ***(R1)***

$f = \sum f_i - (x-1)$ ***A1***

so

$v - e + f + x - 1 = 2x$

$v - e + f = x + 1$ ***A1***

[6 marks]

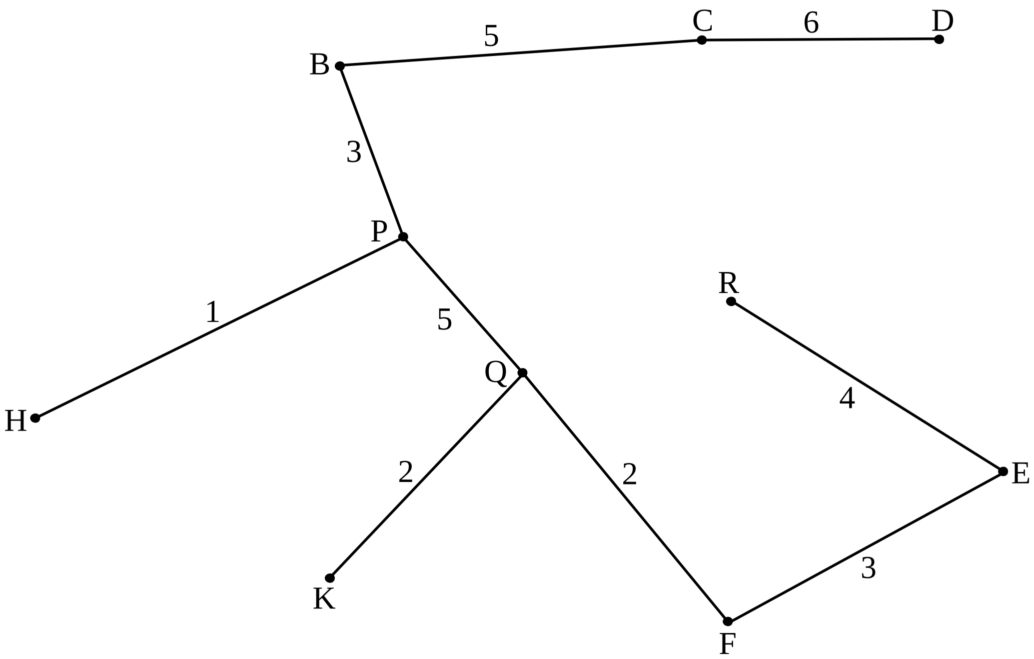
Total [15 marks]

4. (a) (10 vertices so 9 choices)

Choice	Edge	Weight	
1	HP	1	***M1***
=2	KQ	2	
=2	QF	2	
=4	FE	3	
=4	PB	3	
6	ER	4	
=7	PQ	5	
=7	BC	5	
9	CD	6	***A2***

Note: Award ***A1*** for one error and ***A0*** for more than one error.

Total weight $= 31$ ***A1***



A1

[5 marks]

(b) Replacing A gives a lower bound $= 31 + 6 + 11 = 48$ ***M1A1***

[2 marks]

continued ...

Question 4 continued

(c) Given $n \geq 3$ to ensure that cycles are possible, from the starting point there are $(n-1)$ choices for the second vertex, $(n-2)$ choices for the third vertex, *etc*. ***M1A1***

Total number of choices is $(n-1)(n-2)(n-3)\ldots 1 = (n-1)!$ ***A1***

But since each Hamiltonian cycle can be traversed in reverse we need only examine $\frac{(n-1)!}{2}$ cycles. ***R1AG***

[4 marks]

(d) Number of cycles $< \frac{(11-1)!}{2} = 1814400$ (accept $=$ in place of $<$) ***R1***

[1 mark]

Total [12 marks]

5. **EITHER**

By Fermat's little theorem

$n^7 \equiv n \pmod 7$ ***M1***

$n^7 - n$ is divisible by 7 ***A1***

$n^7 - n = n(n^6 - 1)$ ***M1A1***

$= n\ (n^2)^3 - 1\ = n(n^2 - 1)(n^4 + n^2 + 1)$ ***(A1)A1***

$= (n-1)n(n+1)(n^4 + n^2 + 1)$ ***A1***

$6 \mid (n-1)n(n+1)$ because $(n-1)$, n, $(n+1)$ are consecutive integers. ***R2***

As $n^7 - n$ is divisible by 6 and 7 then $42 \mid n^7 - n$. ***R1***

Note: The factors $(n-1)$, n, $(n+1)$ of $(n^7 - n)$ could be obtained by using the remainder theorem.

[10 marks]

OR

By Fermat's little theorem

$n^2 \equiv n \pmod 2$ ***M1A1***

$n^3 \equiv n \pmod 3$ ***A1***

and

$n^7 = (n^3)^2 n \equiv n^2 n \pmod 3 = n^3 \pmod 3 \equiv n \pmod 3$ ***M1A1***

$n^7 = (n^2)^3 n \equiv n^3 n = n^4 = (n^2)^2 \equiv n^2 \pmod 2 \equiv n \pmod 2$ ***M1A1***

but $n^7 \equiv n \pmod 7$ ***R1***

since $42 = 2 \times 3 \times 7$ then $n^7 \equiv n \pmod{42}$ ***R1***

Therefore $42 \mid (n^7 - n)$. ***R1***

[10 marks]