IB DIPLOMA PROGRAMME
PROGRAMME DU DIPLÔME DU BI
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M07/5/MATHL/HP3/ENG/TZ2/XX+

22077209

MATHEMATICS
HIGHER LEVEL
PAPER 3

Wednesday 16 May 2007 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions in one section only.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

Statistics and probability

1. *[Maximum mark: 8]*

(a) The random variable X has a geometric distribution with parameter $p = \frac{1}{4}$.

What is the value of $\mathrm{P}(X \le 4)$? *[3 marks]*

(b) A magazine publisher promotes his magazine by putting a concert ticket at random in one out of every four magazines. If you need 8 tickets to take friends to the concert, what is the probability that you will find your last ticket when you buy the 20th magazine? *[3 marks]*

(c) How are the two distributions in parts (a) and (b) related? *[2 marks]*

2. *[Maximum mark: 14]*

(a) In a random sample of 1100 people in Switzerland it was found that 580 of them had a connection to the Internet. Calculate the 95 % confidence interval for the proportion of people in Switzerland having a connection to the Internet. *[7 marks]*

(b) How large should the sample have been to make the width of the 95 % confidence interval less than 0.02? *[7 marks]*

3. *[Maximum mark: 14]*

Competitors at the Worlds Strongest Man contest have to hold an extremely heavy weight, with their arms held out straight, for as long as possible. It is claimed that a particular training schedule will improve the time, (*i.e.* increase it), that a competitor can hold the weight for. Competitors are tested before and after the training schedule.

The times, in seconds, before and after training are shown in the table below.

Competitor	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Time before training	80	62	45	73	65	53	61	48	81	50	50	29	52	33	71
Time after training	85	74	60	67	69	55	68	46	89	60	64	26	61	33	72

Stating the null and alternative hypotheses carry out an appropriate test at the 1 % significance level to decide if the claim is justified. *[14 marks]*

4. *[Maximum mark: 14]*

(a) If Y has a Poisson distribution $\mathrm{Po}(\mu)$, show that $\mathrm{P}(Y = y+1) = \dfrac{\mu}{y+1}\mathrm{P}(Y = y)$. *[3 marks]*

(b) The number of cars passing a certain point in a road was recorded during 80 equal time intervals and summarized in the table below.

Number of cars	0	1	2	3	4	5
Frequency	4	18	19	20	11	8

Carry out a χ^2 goodness of fit test at the 5 % significance level to decide if the above data can be modelled by a Poisson distribution. *[11 marks]*

5. *[Maximum mark: 10]*

(a) The independent variables U and V are such that $U \sim \mathrm{N}(66, 5)$ and $V \sim \mathrm{N}(19, 3)$. Calculate the probability that a randomly selected observation from U is more than three times a randomly selected observation from V. *[6 marks]*

(b) Let X be a random variable. By expanding the expression $\mathrm{E}\left(X - \mathrm{E}(X)\right)^2$ show that $\mathrm{E}(X^2) \geq \left(\mathrm{E}(X)\right)^2$. *[4 marks]*

SECTION B

Sets, relations and groups

1. *[Maximum mark: 7]*

Show that the function $f:\mathbb{Z}\to\mathbb{Z}$ defined below is a surjection but not an injection.

$$f(x)=\begin{cases}\dfrac{x+2}{2}, \text{ if } x \text{ is even}\\ \dfrac{x+5}{2}, \text{ if } x \text{ is odd}\end{cases}$$ *[7 marks]*

2. *[Maximum mark: 10]*

We define the relation $(x, y)R(p, q)$ if and only if $x^2-y^2=p^2-q^2$ where $(x, y), (p, q)\in\mathbb{R}^2$. Prove that R is an equivalence relation on $\mathbb{R}^2$. Describe geometrically the equivalence class of $(1, 1)$. *[10 marks]*

3. *[Maximum mark: 15]*

a and b are elements of the group G whose binary operation is multiplication.

(a) Use mathematical induction to prove that $(bab^{-1})^n=ba^nb^{-1}$, for all $n\in\mathbb{Z}^+$. *[8 marks]*

(b) Show that $(bab^{-1})^{-1}=ba^{-1}b^{-1}$. *[3 marks]*

(c) Use parts (a) and (b) to show that $(bab^{-1})^n=ba^nb^{-1}$ for all negative integers n. *[4 marks]*

4. *[Maximum mark: 14]*

(a) Show that the set $M = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix}; a,\ b \in \mathbb{R},\ a^2 + b^2 \neq 0 \right\}$ together with matrix multiplication ($\times$) forms a group $\{M, \times\}$. *[6 marks]*

(b) Find an isomorphism from the multiplicative group of non-zero complex numbers to the group $\{M, \times\}$. Justify your answer. *[8 marks]*

5. *[Maximum mark: 6]*

Show that every cyclic group of order greater than or equal to three has at least two generators. *[6 marks]*

6. *[Maximum mark: 8]*

Prove for sets A, B and C that $A \times (B \cup C) = (A \times B) \cup (A \times C)$. *[8 marks]*

SECTION C

Series and differential equations

1. *[Maximum mark: 9]*

Find

(a) $\lim_{s \to 4} \left(\frac{s - \sqrt{3s+4}}{4-s} \right)$ *[4 marks]*

(b) $\lim_{x \to 0} \left(\frac{\cos x - 1}{x \sin x} \right)$ *[5 marks]*

2. *[Maximum mark: 14]*

(a) Sketch on graph paper the slope field for the differential equation $\frac{dy}{dx} = x - y$ at the points (x, y) where $x \in \{0, 1, 2, 3, 4\}$ and $y \in \{0, 1, 2, 3, 4\}$. Use a scale of 2 cm for 1 unit on both axes. *[3 marks]*

(b) On the slope field sketch the curve that passes through the point $(0, 3)$. *[1 mark]*

(c) Solve the differential equation to find the equation of this curve.
Give your answer in the form $y = f(x)$. *[10 marks]*

3. *[Maximum mark: 10]*

(a) Find constants A and B such that $\frac{1}{16r^2 + 8r - 3} = \frac{A}{4r-1} + \frac{B}{4r+3}$. *[4 marks]*

(b) Find an expression for S_n, the nth partial sum of $\sum_{r=1}^{\infty} \left(\frac{1}{16r^2 + 8r - 3} \right)$. *[4 marks]*

(c) Hence show that the series converges. *[2 marks]*

4. *[Maximum mark: 7]*

Find the values of p for which $\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ converges. *[7 marks]*

5. *[Maximum mark: 20]*

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}+(2x-1)\,y=0$ given that $y=2$ when $x=0$.

(a) (i) Show that $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}=(1-2x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-6\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

(ii) By finding the values of successive derivatives at $x=0$ obtain a Maclaurin series for y up to and including the x^4 term. *[10 marks]*

(b) A local maximum value of y occurs when $x=0.5$. Use your series to calculate an approximation to this maximum value. *[2 marks]*

(c) Use Euler's method with a step value of 0.1 to obtain a second approximation for the maximum value of y. Set out your solution in tabular form. *[6 marks]*

(d) How can each of the approximations found in (b) and (c) be made more accurate? *[2 marks]*

SECTION D

Discrete mathematics

1. *[Maximum mark: 6]*

Show that $(x+y)^p \equiv (x^p + y^p) \pmod{p}$ where $x,\ y \in \mathbb{Z}^+$ and p is a prime number. *[6 marks]*

2. *[Maximum mark: 17]*

(a) Use Euclid's algorithm to find $\gcd(858,\ 714)$ and hence write the gcd as a linear combination of 858 and 714. *[12 marks]*

(b) A lattice point $(x,\ y)$ is a point on the Cartesian plane where $x,\ y \in \mathbb{Z}$. Find all the lattice points through which the line $5x + 8y = 1$ passes. *[5 marks]*

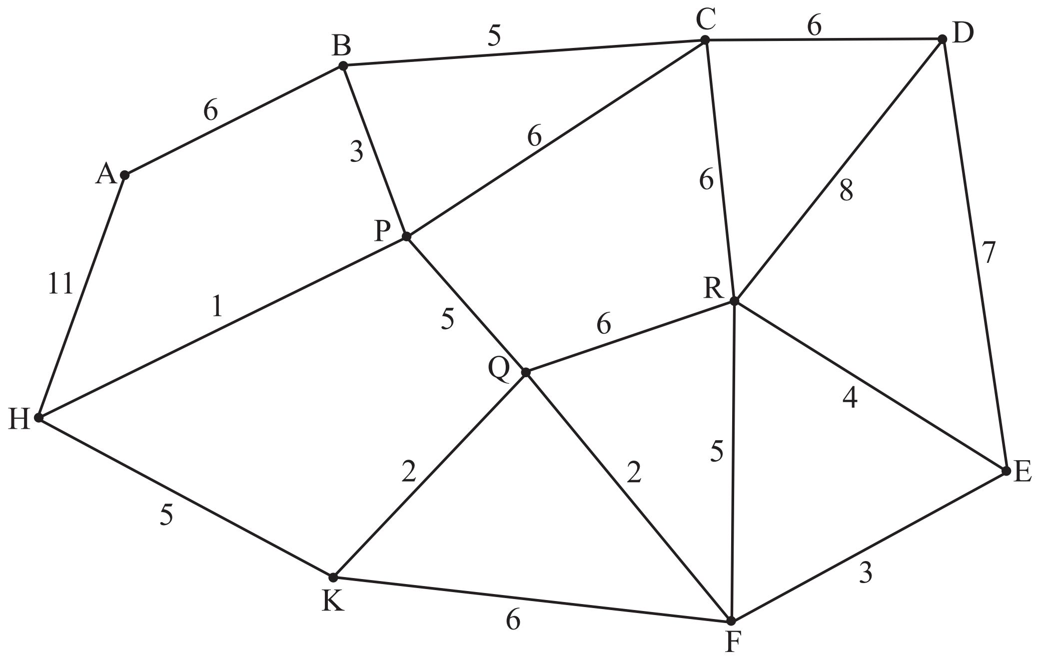
3. *[Maximum mark: 15]*

(a) Draw the graph $K_{3,\,3}$. *[3 marks]*

(b) Prove that $K_{3,\,3}$ is not planar. *[6 marks]*

(c) With the usual notation, Euler's formula $v - e + f = 2$ applies to connected planar graphs. Find a formula connecting v, e and f for a non-connected planar graph that has x components. *[6 marks]*

4. *[Maximum mark: 12]*



The weighted graph G is shown above. Graph G' is produced by deleting vertex A from G.

(a) Use Kruskal's algorithm to find the minimum spanning tree of graph G' and state its weight. *[5 marks]*

(b) Hence find the weight of a lower bound for the Hamiltonian cycle in G beginning at vertex A. *[2 marks]*

(c) Prove for a **complete** graph with n vertices, that no more than $\frac{(n-1)!}{2}$, $n \geq 3$ Hamiltonian cycles have to be examined to find the Hamiltonian cycle of least weight. *[4 marks]*

(d) How many cycles in G would have to be examined to find the one with the least weight? *[1 mark]*

5. *[Maximum mark: 10]*

Prove that $42 \mid (n^7 - n)$ where $n \in \mathbb{Z}^+$. *[10 marks]*