MARKSCHEME

November 2006

MATHEMATICS

Higher Level

Paper 3

19 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**: often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown (or working which gains no other marks).

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working, (or working which gains no other marks).*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.
- For consistency within the markscheme, ***N*** marks are noted for every part, even when these match the mark breakdown. In these cases, the marks may be recorded in either form *e.g.* ***A2*** or ***N2***.

4 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). Apply a **MR** penalty of 1 mark to that question. Award the marks as usual and then write –1(**MR**) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalised only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled **(d)** and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc.*
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

Candidates should be penalized ***once only IN THE PAPER*** *for an accuracy error* ***(AP)***. *Award the marks as usual then write* ***(AP)*** *against the answer. On the* ***front*** *cover write –1****(AP).*** *Deduct 1 mark from the total for the paper, not the question*.

- If a final correct answer is incorrectly rounded, apply the ***AP***.
- If the level of accuracy is not specified in the question, apply the ***AP*** for correct answers not given to three significant figures.

If there is no working shown, and answers are given to the correct two significant figures, apply the ***AP***. However, do **not** accept answers to one significant figure without working.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Examples

Exemplar material is available under examiner training on http://courses.triplealearning.co.uk. Please refer to this material before you start marking, and when you have any queries. Please also feel free to contact your Team Leader if you need further advice.

SECTION A

Statistics and probability

1. (a) H_0 : Mean loss $= 5$ kg *A1*
H_1 : Mean loss < 5 kg *A1*

[2 marks]

(b) The losses are
6.8 4.7 3.9 4.3 2.0 5.9 3.6 3.3 2.8 4.2 *(M1)(A1)*
Using a one-sided *t*-test on difference between weights before and after
$t = 1.90$ (accept $+$ or $-$) *A1A1*
p-value $= 0.0447$ *A2* *N4*

(i) At 1%, insufficient evidence to reject H_0 (critical value 2.821) *A1*

(ii) At 10%, reject H_0 (critical value 1.383) *A1*

[8 marks]

(c) The differences are a random sample from a normal distribution. *R1*

[1 mark]

Total [11 marks]

2. (a) $\bar{x} = 12.6$ *(A1)*
$\text{SE} = \dfrac{2.5}{5} = 0.5$ *(A1)*
90% confidence limits are using 1.645 *(A1)*
$12.6 \pm 1.645 \times 0.5$ *M1A1*
giving $[11.8, 13.4]$ *A1* *N6*

[6 marks]

(b) $P(X \le 14) = \Phi\left(\dfrac{14-\mu}{2.5}\right) = 0.55$ *M1A1*
$\dfrac{14-\mu}{2.5} = 0.1256...$ *M1A1*
$\mu = 13.7$ *A1*
Not consistent because outside confidence interval. *A1*

[6 marks]

Total [12 marks]

3. The hypotheses are

H_0 : The model fits the data ***A1***

H_1 : The model does not fit the data ***A1***

Expected frequencies are given by $\mathrm{E}(X = x) = 420 \times \frac{x}{21}$ ***(M1)(A1)***

x	f_o	f_e
1	25	20
2	46	40
3	64	60
4	82	80
5	99	100
6	104	120

A4

$\chi^2 = \frac{5^2}{20} + \frac{6^2}{40} + \frac{4^2}{60} + \frac{2^2}{80} + \frac{1^2}{100} + \frac{16^2}{120}$ ***M1A1***

$= 4.61$ ***A1***

$\mathrm{DF} = 5$ ***A1***

Critical value $= 11.07$ at the $5\ \%$ level. ***A1***

We conclude that the claim is accepted. ***R1***

[14 marks]

4. (a) $X \sim \mathrm{B}(20, 0.15)$ ***(M1)***

Using gdc, $\mathrm{P}(X \geq 5) = 0.170$ ***A2*** ***N2***

[3 marks]

(b) By trial and error around sensible values of n, ***M1***

$\mathrm{P}(X \geq 5 \mid 30 \text{ bought}) = 0.476$ ***A1***

$\mathrm{P}(X \geq 5 \mid 31 \text{ bought}) = 0.506$ ***A1***

$\therefore$ Minimum number $= 31$ ***A1*** ***N4***

[4 marks]

(c) (i) $\mathrm{P}(X = x) = \binom{x-1}{4} 0.15^5 \times 0.85^{x-5}$ ***A2***

(ii) X is negative binomial with $r = 5$, $p = 0.15$ ***(A1)***

$\mathrm{E}(X) = \frac{r}{p} = 33.3$ ***M1A1***

(iii) $$\frac{\mathrm{P}(X = x)}{\mathrm{P}(X = x-1)} = \frac{\binom{x-1}{4} 0.15^5 \; 0.85^{x-5}}{\binom{x-2}{4} 0.15^5 \; 0.85^{x-6}}$$ ***M1A1***

$$= \frac{(x-1)!}{(x-5)!} \frac{(x-6)!}{(x-2)!} 0.85$$ ***M1A1***

$$= \frac{0.85(x-1)}{x-5}$$ ***AG***

(iv) $\mathrm{P}(X = x) > \mathrm{P}(X = x-1)$ as long as $\frac{0.85(x-1)}{x-5} > 1$ ***M1A1***

$x < \frac{4.15}{0.15}$ ***A1***

$= \frac{83}{3}$ ***AG***

It follows that $p_{27} > p_{26}$ ***M1A1***

but $p_{28} \leq p_{27}$ ***A1***

It follows that the most probable value of X is 27. ***R1***

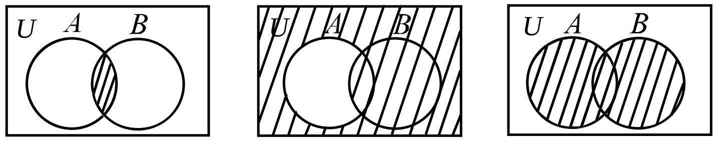
[16 marks]

Total [23 marks]

SECTION B

Sets, relations and groups

1. (a)



C D E

A1A1A1

[3 marks]

(b) $D' \cup C$

$= (A' \cup B)' \cup (A \cap B)$ ***M1A1***

$= (A \cap B') \cup (A \cap B)$ ***A1***

$= A$ ***AG***

[3 marks]

(c) $D \cap E$

$= (A' \cup B) \cap (A \cup B)$ ***M1***

$= (A' \cap A) \cup (A' \cap B) \cup (B \cap A) \cup (B \cap B)$ ***A1***

$= B \cup (A' \cap A)$ ***(A1)***

$= B \cup \varnothing$

$= B$ ***AG***

[3 marks]

Total [9 marks]

2. (a) (i)

$\bullet_8$	**1**	**3**	**5**	**7**
1	1	3	5	7
3	3	1	**7**	5
5	**5**	7	1	3
7	7	5	3	**1**

G ***A4***

> **Note:** Award ***A3*** for 1 error, ***A2*** for 2 errors and ***A0*** for 3 or more errors.

(ii)

$*_{15}$	**3**	**6**	**9**	**12**
3	**9**	3	**12**	6
6	3	6	9	12
9	12	9	6	3
12	6	12	**3**	9

H ***A2***

> **Note:** Award ***A1*** for 1 error, ***A0*** for 2 or more errors.

[6 marks]

(b) *H*: Identity is 6: order of 3 is 4 ***A2***
order of 9 is 2
order of 12 is 4

G: Identity is 1: order of 3 is 2 ***A1***
order of 5 is 2
order of 7 is 2

The orders of *G* and *H* are not comparable. ***(M1)***
Hence the two groups are not isomorphic. ***A1***

[5 marks]

Total [11 marks]

3. (a) R_1 is not an equivalence relation ***A1***
because it is not reflexive (*aRa* does not hold). ***R1***

R_2 is not an equivalence relation ***A1***
because it is not symmetric ***R1***
BRA is true whereas *ARB* does not hold. ***A1***

R_3 is not an equivalence relation ***A1***
because it is not transitive ***R1***
dRf and *fRe* are true, whereas *dRe* does not hold. ***A2***

Note: For all 3 conditions, accept any counter example.

R_4 is an equivalence relation ***A1***
because it is reflexive (there are only 1s on the diagonal), ***A1***
symmetric (the table is symmetric with respect to the diagonal) ***A1***
transitive (all the 0 on the square of the matrix are also 0 on the matrix) ***A3***
(other method: checking all the cases).

[15 marks]

(b) The equivalence classes are $\{D, F\}$ and $\{E, G\}$. ***A1A1***

[2 marks]

Total [17 marks]

4. (a) Range of f is $y > 2$ therefore f is not surjective on $\mathbb{R}^+$ ***R2***

[2 marks]

(b) If $g(a, b) = g(c, d)$

then $3a + 2b = 3c + 2d$ (1)

and $2a + b = 2c + d$ (2) ***M1***

$(1) - 2\times(2)$ gives $a = c$ ***A1***

hence $b = d$ ***A1***

therefore g is injective

let $g(x, y) = (u, v)$

then $u = 3x + 2y$

$v = 2x + y$ ***M1***

solving simultaneous equations gives

$x = -u + 2v$ ***A1***

$y = 2u - 3v$ ***A1***

therefore g is surjective ***R1***

and hence $g^{-1}(u, v) = (-u + 2v, 2u - 3v)$ ***A2***

[9 marks]

(c) (i) h is not injective ***A1***

because, (x, y) and (y, x) have the same image, $x \neq y$ ***R1***

(ii) h is not surjective ***A1***

Any counter example satisfying $\left\{ (x, y) \,\middle|\, x^2 < 4y \right\}$ ***A2***

[5 marks]

Total [16 marks]

5. Let e be the identity element of G.

Let $a, b \in G$ ***M1***

Then $a^2 = e$

$b^2 = e$ ***A1***

$abab = e$ ***A1***

Hence $a(abab)b = aeb$ ***M1A1***

$a^2bab^2 = ab$ ***A1***

$ba = ab$ ***A1***

So G is abelian ***AG***

[7 marks]

SECTION C

Series and differential equations

1. (a) $\dfrac{u_{n+1}}{u_n} = \dfrac{n!}{(n+1)!}$ ***M1A1***

$= \dfrac{1}{n+1} \to 0$ as $n \to \infty$ ***A1A1***

Therefore series convergent by ratio test. ***AG***

[4 marks]

(b) $\dfrac{2^n}{n!} < 1$, $\forall n > 3$ ***(M1)***

$2^n < n!$, $\forall n > 3$

$1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \ldots < 1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots$ ***M1A1***

so $S_\infty < \dfrac{1}{1 - \frac{1}{2}}$ $(= 2)$ ***A1***

that is $S_\infty < 2$ ***AG***

[4 marks]

(c) $\mathrm{e} - 1$ ***A1***

[1 mark]

Total [9 marks]

2. (a) Putting $f(x) = \ln x$

$f(1) = 0$ ***A1***

$f'(x) = \frac{1}{x},\ f'(1) = 1$ ***A1***

$f''(x) = -\frac{1}{x^2},\ f''(1) = -1$ ***A1***

$f'''(x) = \frac{2}{x^3},\ f'''(1) = 2$ ***A1***

$\ln x \approx (x-1) + \frac{(x-1)^2}{2} \times (-1) + \frac{(x-1)^3}{6} \times 2$ ***M1A1***

$= x - 1 - \frac{1}{2}(x^2 - 2x + 1) + \frac{1}{3}(x^3 - 3x^2 + 3x - 1)$ ***A1***

$= \frac{x^3}{3} - \frac{3x^2}{2} + 3x - \frac{11}{6}$ ***AG***

[7 marks]

(b) Integrating both sides of the above result

$x \ln x - x \approx \frac{x^4}{12} - \frac{x^3}{2} + \frac{3x^2}{2} - \frac{11x}{6} + C$ ***M1A1***

Putting $\ln 1 = 0$ gives $C = -\frac{1}{4}$ so ***(M1)A1***

$\ln x \approx \frac{x^3}{12} - \frac{x^2}{2} + \frac{3x}{2} - \frac{5}{6} - \frac{1}{4x}$ ***A2***

[6 marks]

(c) When $x = 1.5$

$\ln x = 0.4054...$ ***A1***

$\frac{x^3}{3} - \frac{3x^2}{2} + 3x - \frac{11}{6} = 0.4166...$ ***A1***

$\frac{x^3}{12} - \frac{x^2}{2} + \frac{3x}{2} - \frac{5}{6} - \frac{1}{4x} = 0.4062...$ ***A1***

The second approximation is nearer the time value and therefore better. ***A1***

[4 marks]

Total [17 marks]

3. (a) **METHOD 1**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(\frac{1+x}{1-x}\right)\right)=\frac{(1-x)}{(1+x)}\frac{(1-x+1+x)}{(1-x)^2}$$ ***M1A2***

$$=\frac{2}{1-x^2}$$ ***AG***

METHOD 2

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(1+x)-\ln(1-x)\right)=\frac{1}{1+x}+\frac{1}{1-x}$$ ***M1A2***

$$=\frac{2}{1-x^2}$$ ***AG***

[3 marks]

(b) Put $y=vx$ so that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=v+x\dfrac{\mathrm{d}v}{\mathrm{d}x}$ ***(M1)A1***

Substituting

$$x^2\left(v+x\frac{\mathrm{d}v}{\mathrm{d}x}\right)=x^2(1+v-v^2)$$ ***(M1)(A1)***

$$x\frac{\mathrm{d}v}{\mathrm{d}x}=1-v^2$$ ***A1***

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}v}{1-v^2}$$ ***A1***

$$\ln x=\int\frac{\mathrm{d}v}{1-v^2}+C$$ ***M1A1***

$$=\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)+C$$ ***A1***

Substituting $\left(1,\dfrac{1}{2}\right)$

$$0=\frac{1}{2}\ln 3+C \text{ giving } C=-\frac{1}{2}\ln 3$$ ***(M1)A1***

$$\ln x^2=\ln\left(\frac{x+y}{3(x-y)}\right)$$ ***M1A1***

$$3x^2=\frac{x+y}{x-y}$$ ***A1***

giving $y=\dfrac{3x^3-x}{3x^2+1}$ ***A2***

[16 marks]

Total [19 marks]

4. (a) (i) $\displaystyle\int_{-n}^{\alpha n}\frac{x\,\mathrm{d}x}{1+x^2}=\frac{1}{2}\Big[\ln(1+x^2)\Big]_{-n}^{\alpha n}$ ***M1A1***

$\displaystyle=\frac{1}{2}\ln\left(\frac{1+\alpha^2n^2}{1+n^2}\right)$ ***M1A1***

(ii) $\displaystyle\to\frac{1}{2}\ln\alpha^2$ or $\ln\alpha$ as $n\to\infty$ ***M1A1***

[6 marks]

(b) Limit $\displaystyle=\lim_{x\to 0}\left(\frac{\beta\sec^2\beta x-\beta\sec^2 x}{\beta\cos\beta x-\beta\cos x}\right)$ ***M1A1A1***

$\displaystyle=\lim_{x\to 0}\left(\frac{2\beta\sec^2\beta x\tan\beta x-2\sec^2 x\tan x}{-\beta\sin\beta x+\sin x}\right)$ ***A1A1***

$\displaystyle=\lim_{x\to 0}\left(\frac{2\beta^2\sec^4\beta x+4\beta^2\sec^2\beta x\tan^2\beta x-2\sec^4 x-4\sec^2 x\tan^2 x}{-\beta^2\cos\beta x+\cos x}\right)$ ***A1A1***

$\displaystyle=\frac{2\beta^2-2}{-\beta^2+1}$ ***A1***

$=-2$ ***A1***

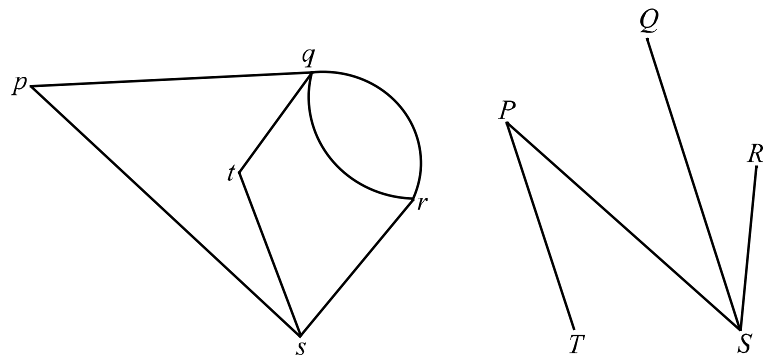
[9 marks]

Total [15 marks]

SECTION D

Discrete mathematics

1. (a)



G_1 G_2 ***A2A2***

[4 marks]

(b) (i) G_1 is not simple (some entries are "2"s) ***R1***
G_2 is simple (no entries on the diagonal and no double edge) ***R1***

(ii) G_1 is connected
(p connected to q connected to r connected to s connected to t) ***R1***

G_2 is connected (E connected to H connected to F and H connected to G;
and E connected to K) ***R1***

(iii) G_1 is bipartite (take $\{p, r, t\}$ and $\{q, s\}$) ***M1A1***

G_2 is bipartite (take $\{P, R, Q\}$ and $\{T, S\}$) ***M1A1***

(iv) G_1 is not a tree (it contains two edges)

$$G_2 \text{ is a tree } \begin{array}{ccccc} P & \rightarrow & S & \rightarrow & Q \\ \downarrow & & \downarrow & & \\ T & & R & & \end{array}$$ ***R1R1***

(v) G_1 contains a Eulerian trail (only two vertices have an odd degree) ***M1***

Example: $r \to q \to p \to s \to r \to q \to t \to s$ ***A1***

G_2 does not contain an Eulerian trail (it has more than two vertices of odd degree) ***M1A1***

[14 marks]

Total [18 marks]

2. This is equivalent to solving $-19x+13y=4$ ***(R1)***

Using Euclid's algorithm $19=13+6$ ***M1A1***

$13=2\times6+1$ ***A1***

Hence $1=13-2\times6=13-2(19-13)$ ***(A1)***

$=3\times13-2\times19$ ***A1***

Therefore $-19\times8+13\times12=4$ ***A1***

The general solution is $x=8-13k$

$y=12-19k \quad k\in\mathbb{Z}$ ***M1A2***

[10 marks]

3. (a) There are 7 vertices so 6 choices must be made ***(A1)***

Edge	Weight	Choice	
BG	1	1st	***M2A1***
EF	1	2nd	
ED	2	3rd	***A1***
AB	4	4th	***A1***
BC	5	5th	***A1***
GC	5	Reject	
DF	5	Reject	***A1***
CF	6	6th	***A1***

[9 marks]

(b) Total weight $1+1+2+4+5+6=19$ ***A2***

[2 marks]

Total [11 marks]

4. (a) If a graph is not simple then it must contain either a loop or a double edge and therefore contains a cycle so that it cannot be a tree. Hence if it is a tree it must be simple. ***M1A1R1***

[3 marks]

(b) (i) Let *A* and *B* be the two subsets of vertices of the bipartite graph *G*. Then in the complement of *G*, *A* will have no edge linking any of its elements to *B*. Hence the complement of *G* is not connected since it contains two unconnected subgraphs. ***M1A2***

(ii) Let *H* be the graph consisting of three vertices and no edge. Its complement is obviously not bipartite. ***A2***

[5 marks]

Total [8 marks]

5. Since p divides $x^p + y^p$, $x^p - x$ and $y^p - y$,

then p divides $(x - x^p) + (y - y^p) + (x^p + y^p) = x + y$ ***R2A1***

So $x + y = kp$ where k is an integer ***M1A1***

and $y = kp - x$ ***M1A1***

Therefore $x^p + y^p = x^p + (kp - x)^p = x^p + Ap^2 + pkpx^{p-1} - x^p$

where A is an integer. ***M1A3***

So $x^p + y^p = (A + kx^{p-1})p^2$ and therefore is divisible by p^2. ***M1A1***

[13 marks]