



**MATHEMATICS
HIGHER LEVEL
PAPER 3**

Monday 15 May 2006 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions in one section only.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

Statistics and probability

1. [Maximum mark: 8]

A drug company claims that a new drug cures 75 % of patients suffering from a certain disease. However, a medical committee believes that less than 75 % are cured. To test the drug company's claim, a trial is carried out in which 100 patients suffering from the disease are given the new drug. It is found that 68 of these patients are cured.

- (a) State suitable hypotheses. [2 marks]
- (b) Find the p -value of your test. [4 marks]
- (c) State your conclusion using a significance level of
- (i) 10 %;
- (ii) 1 %. [2 marks]

2. [Maximum mark: 12]

A scientific expedition discovers a large colony of birds. The weights x kg of a random sample of 200 of these birds are measured and the following results obtained:

$$\sum x = 224.4, \quad \sum (x - \bar{x})^2 = 5.823$$

- (a) Calculate unbiased estimates of the mean μ and the variance σ^2 of the weights of these birds. [4 marks]
- (b) Find a 95 % confidence interval for μ . [6 marks]
- (c) State, with a reason, whether or not your answer requires the assumption that the weights are normally distributed. [2 marks]

3. [Maximum mark: 9]

Sarah cycles to work and she believes that the mean time taken to complete her journey is 30 minutes. To test her belief, she records the times (in minutes) taken to complete her journey over a 10-day period as follows:

30.1 32.3 33.6 29.8 28.9 30.6 31.1 30.2 32.1 29.4

You may assume that the journey times are normally distributed with mean μ minutes.

- (a) State suitable hypotheses. [2 marks]
- (b) Test Sarah’s belief, at the 5 % significance level. [5 marks]
- (c) Justify your choice of test. [2 marks]

4. [Maximum mark: 14]

Buses arrive at a bus-stop T minutes apart, where T may be assumed to have an exponential distribution with probability density function

$$f(t) = \frac{1}{10} e^{-\frac{t}{10}} \text{ for } t \geq 0.$$

- (a) Show that
 - (i) $P(T > t) = e^{-\frac{t}{10}}$;
 - (ii) $P(T \leq t + s | T > t) = 1 - e^{-\frac{s}{10}}$, where $s > 0$. [10 marks]
- (b) Bill arrives at the bus-stop five minutes after the previous bus arrived at the bus-stop. Find the probability that the next bus arrives within 10 minutes of his arrival at the bus-stop. [4 marks]

5. [Maximum mark: 17]

The random variable X is thought to have a geometric distribution with probability mass function

$$P(X = x) = p(1 - p)^{x-1} \text{ for } x \in \mathbb{Z}^+,$$

where p is an unknown parameter.

The value of X is recorded on 100 independent occasions with the following results.

x	Frequency
1	45
2	26
3	16
4	10
5 or more	3

(a) (i) Calculate the mean of these data.

(ii) Deduce that the estimated value of p is $\frac{1}{2}$. [4 marks]

(b) Calculate an appropriate value of χ^2 . Test, at the 5 % significance level, whether or not these data can be modelled by a geometric distribution. [13 marks]

SECTION B

Sets, relations and groups

1. [Maximum mark: 15]

The function f is defined by

$$f : \mathbb{R} \rightarrow \mathbb{R} \text{ where } f(x) = e^{\sin x} - 1.$$

(a) Find the exact range, A , of f . [3 marks]

(b) (i) Explain why f is not an injection.

(ii) Giving a reason, state whether or not f is a surjection. [4 marks]

(c) The function g is now defined to be $g : [-k, k] \rightarrow A$, where $g(x) = e^{\sin x} - 1$ and $k > 0$.

(i) Find the maximum value of k for which g is an injection.

For this value of k ,

(ii) find an expression for $g^{-1}(x)$;

(iii) write down the domain of g^{-1} . [8 marks]

2. [Maximum mark: 15]

Let $S = \{2, 4, 6, 8, 10, 12, 14\}$. The relation R is defined on S such that for $a, b \in S$, $a R b$ if and only if $a^2 \equiv b^2 \pmod{6}$.

(a) Show that R is an equivalence relation. [6 marks]

(b) Find all the equivalence classes. [9 marks]

3. [Maximum mark: 7]

Consider the binary operation a divided by b defined on \mathbb{R}^+ . Determine whether or not each of the four group axioms is satisfied. [7 marks]

4. [Maximum mark: 15]

Consider the group G defined on the set $S = \{1, 2, 4, 5, 7, 8\}$ having the following Cayley table.

*	1	2	4	5	7	8
1	1	2	4	5	7	8
2	2	4	8	1	5	7
4	4	8	7	2	1	5
5	5	1	2	7	8	4
7	7	5	1	8	4	2
8	8	7	5	4	2	1

(a) Explain what is meant by saying that this table is a Latin square. [1 mark]

(b) Solve the equation

$$2 * x * 7 = 4 \text{ where } x \in S. \quad [4 \text{ marks}]$$

(c) (i) Show that G is cyclic and find the generators.

(ii) List the proper subgroups of G . [10 marks]

5. [Maximum mark: 8]

Suppose that G is a group and H is a non-empty subset of G . Show that if $ab^{-1} \in H$ whenever $a, b \in H$ then H is a subgroup of G . [8 marks]

SECTION C

Series and differential equations

1. [Maximum mark: 9]

Given that $\frac{dy}{dx} = \frac{y+2}{xy+1}$ and $y=1$ when $x=0$, use Euler's method with interval $h=0.5$ to find an approximate value of y when $x=1$. [9 marks]

2. [Maximum mark: 12]

(a) Show that $\int \tan x \, dx = \ln \sec x + C$, where C is a constant. [2 marks]

(b) Hence find an integrating factor for solving the differential equation

$$\frac{dy}{dx} + y \tan x = \sec x. \quad [2 \text{ marks}]$$

(c) Solve this differential equation given that $y=2$ when $x=0$.
Give your answer in the form $y = f(x)$. [8 marks]

3. [Maximum mark: 9]

Find the value of

(a) $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$; [3 marks]

(b) $\lim_{x \rightarrow 0} x \ln x$. [6 marks]

4. [Maximum mark: 15]

(a) (i) Given that $\sum_{n=1}^{\infty} u_n$ is convergent, where $u_n \geq 0$, prove that $\sum_{n=1}^{\infty} u_n^2$ is also convergent.

(ii) State, with a reason, whether or not the converse of this result is true. [5 marks]

(b) Use the integral test to determine the set of values of k for which the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^k}$$

(i) is convergent;

(ii) is divergent. [10 marks]

5. [Maximum mark: 15]

Consider the function f defined by

$$f(x) = \arcsin x, \text{ for } |x| \leq 1.$$

The derivatives of $f(x)$ satisfy the equation

$$(1-x^2)f^{(n+2)}(x) - (2n+1)x f^{(n+1)}(x) - n^2 f^{(n)}(x) = 0, \text{ for } n \geq 1.$$

The coefficient of x^n in the Maclaurin series for $f(x)$ is denoted by a_n . You may assume that the series contains only odd powers of x .

(a) (i) Show that, for $n \geq 1$, $(n+1)(n+2)a_{n+2} = n^2 a_n$.

(ii) Given that $a_1 = 1$, find an expression for a_n in terms of n , valid for odd $n \geq 3$. [7 marks]

(b) Find the radius of convergence of this Maclaurin series. [4 marks]

(c) Find an approximate value for π by putting $x = \frac{1}{2}$ and summing the first three non-zero terms of this series. Give your answer to **four** significant figures. [4 marks]

SECTION D**Discrete mathematics**

1. [Maximum mark: 9]

- (a) Convert the number 95 from base 10 to base 6. [3 marks]
- (b) Working in base 6, square your answer to part (a). [4 marks]
- (c) Convert your answer to part (b) to a base 10 number. [2 marks]

2. [Maximum mark: 8]

Consider the diophantine equation

$$\lambda x - 2y = 1, \text{ where } \lambda \in \mathbb{Z}.$$

- (a) Explain briefly why this equation has no solution when $\lambda = 4$. [2 marks]
- (b) Find the general solution to this equation when $\lambda = 3$. [6 marks]

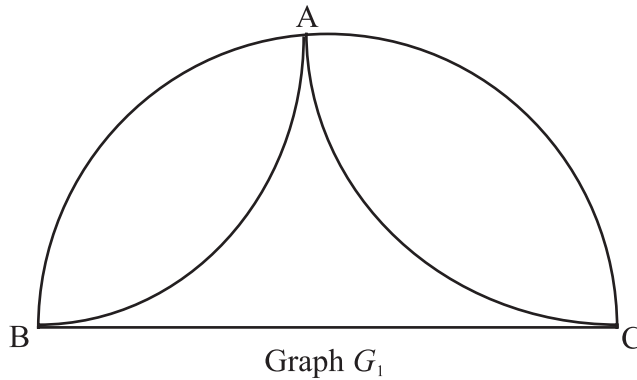
3. [Maximum mark: 16]

- (a) Show that the sum of the degrees of all the vertices of a graph is even. [3 marks]
- (b) There are nine men at a party. By considering an appropriate graph, show that it is impossible for each man to shake hands with exactly five other men. [4 marks]
- (c) For a connected planar graph, prove Euler's relation, $v - e + f = 2$. [9 marks]

4. [Maximum mark: 16]

The graphs G_1 , G_2 and G_3 are illustrated below.

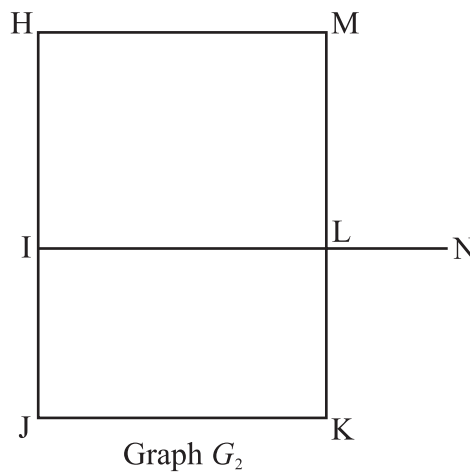
(a)



Write down the adjacency matrix, A_{G_1} , of G_1 . Evaluate $A_{G_1}^2$ and **hence** state the number of paths of length 2 beginning and ending at C.

[6 marks]

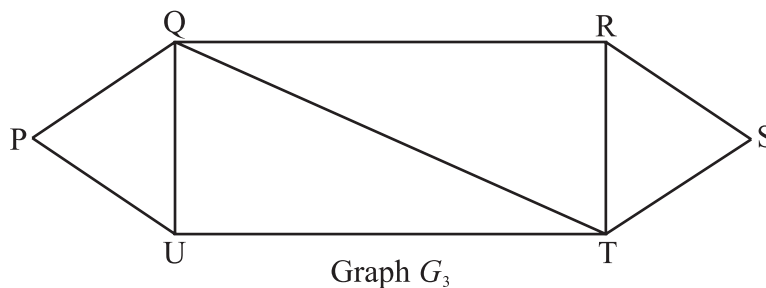
(b)



Determine whether or not G_2 is bipartite.

[4 marks]

(c)



Explain briefly why G_3 does not have an Eulerian circuit. Redraw G_3 and add an edge so that the resulting graph does have an Eulerian circuit. Write down an Eulerian circuit.

[6 marks]

5. [Maximum mark: 11]

The weights of the edges of a graph with vertices A, B, C, D and E are given in the following table.

	A	B	C	D	E
A	-	10	15	11	16
B	10	-	12	19	13
C	15	12	-	18	14
D	11	19	18	-	17
E	16	13	14	17	-

- (a) Use any method to find an upper bound for the travelling salesman problem for this graph. *[2 marks]*

 - (b) (i) Use Kruskal’s algorithm to find and draw a minimum spanning tree for the subgraph obtained by removing the vertex E from the graph.

 - (ii) State the total weight of this minimum spanning tree and hence find a lower bound for the travelling salesman problem for this graph. *[9 marks]*
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