MARKSCHEME

November 2005

MATHEMATICS

Higher Level

Paper 2

30 pages

Instructions to Examiners

Note: **Where there are two marks (*e.g. M2*, *A2*) for an answer do not split the marks unless otherwise instructed.**

1 Method of marking

(a) All marking must be done using a **red** pen.

(b) Marks should be noted on candidates' scripts as in the markscheme:

- show the breakdown of individual marks using the abbreviations ***(M1)***, ***(A2)*** *etc.*, unless a part is completely correct;
- write down each part mark total, indicated on the markscheme (for example, ***[3 marks]***) – it is suggested that this be written at the end of each part, and underlined;
- write down and circle the total for each question at the end of the question.

2 Abbreviations

The markscheme may make use of the following abbreviations:

(M) Marks awarded for **Method**

(A) Marks awarded for an **Answer** or for **Accuracy**

(N) Marks awarded for correct answers, if **no** working (or no relevant working) shown: they may not necessarily be all the marks for the question. Examiners should only award these marks for correct answers where there is no working.

(R) Marks awarded for clear **Reasoning**

(AG) **Answer Given** in the question and consequently marks are **not** awarded

Note: Unless otherwise stated, it is not possible to award ***(M0)(A1).***

Follow through (ft) marks should be awarded where a correct method has been attempted but error(s) are made in subsequent working which is essentially correct.
- Penalize the error when it first occurs
- Accept the incorrect result as the appropriate quantity in all subsequent working
- If the question becomes much simpler then use discretion to award fewer marks

Examiners should use **(d)** to indicate where discretion has been used. It should only be used for decisions on follow through and alternative methods. It must be accompanied by a brief note to explain the decision made

3 Using the Markscheme

(a) This markscheme presents a particular way in which each question may be worked and how it should be marked. **Alternative methods** have not always been included. Thus, if an answer is wrong then the working must be carefully analysed in order that marks are awarded for a different method in a manner which is consistent with the markscheme. Indicate the awarding of these marks by (**d**).

Where alternative methods for complete questions or parts of questions are included, they are indicated by **METHOD 1**, **METHOD 2**, *etc*. Other alternative part solutions are indicated by **EITHER**….**OR.** It should be noted that ***G*** marks have been removed, and GDC solutions will not be indicated using the **OR** notation as on previous markschemes.

Candidates are expected to show working on this paper, and examiners should not award full marks for just the correct answer. Where it is appropriate to award marks for correct answers with no working (or no relevant working), it will be shown on the markscheme using the ***N*** notation. All examiners will be expected to award marks accordingly in these situations.

(b) Unless the question specifies otherwise, accept **equivalent forms**. For example: $\frac{\sin\theta}{\cos\theta}$ for $\tan\theta$. On the markscheme, these equivalent numerical or algebraic forms will generally be written in brackets after the required answer. Paper setters will indicate the required answer, by allocating full marks at that point. Further working should be ignored, even if it is incorrect. For example: if candidates are asked to factorize a quadratic expression, and they do so correctly, they are awarded full marks. If they then continue and find the roots of the corresponding equation, do not penalize, even if those roots are incorrect, *i.e.* once the correct answer is seen, ignore further working.

(c) As this is an international examination, all **alternative forms of notation** should be accepted. For example: 1.7 , 1·7, 1,7; different forms of vector notation such as $\vec{u}$, $\bar{u}$, $\underline{u}$; $\tan^{-1} x$ for arctan x.

4 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

There are two types of accuracy error. Candidates should be penalized **once only IN THE PAPER** for an accuracy error **(AP).**

Award the marks as usual then write –1(**AP**) against the answer and also on the **front** cover

Rounding errors: only applies to final answers not to intermediate steps.

Level of accuracy: when this is not specified in the question the general rule *unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures* applies.

- If a final correct answer is incorrectly rounded, apply the **AP**
  OR
- If the level of accuracy is not specified in the question, apply the **AP** for answers not given to 3 significant figures. (Please note that this has changed from 2003).

Note: **If there is no working shown**, and answers are given to the correct two significant figures, apply the **AP**. However, do **not** accept answers to one significant figure without working.

5 Graphic Display Calculators

Many candidates will be obtaining solutions directly from their calculators, often without showing any working. They have been advised that they must use mathematical notation, not calculator commands when explaining what they are doing. Incorrect answers without working will receive no marks. However, if there is written evidence of using a graphic display calculator correctly, method marks may be awarded. Where possible, examples will be provided to guide examiners in awarding these method marks.

Examples

1. Accuracy

A question leads to the answer 4.6789….

- 4.68 is the correct 3 s.f. answer.
- 4.7, 4.679 are to the wrong level of accuracy : both should be penalised the first time this type of error occurs.
- 4.67 is incorrectly rounded – penalise on the first occurrence.

Note: All these "incorrect" answers may be assumed to come from 4.6789..., even if that value is not seen, but previous correct working is shown. However, 4.60 is wrong, as is 4.5, 4.8, and these should be penalised as being incorrect answers, not as examples of accuracy errors.

2. Alternative solutions

The points P, Q, R are three markers on level ground, joined by straight paths PQ, QR, PR as shown in the diagram. $\text{QR} = 9\text{ km}$, $\text{P}\hat{\text{Q}}\text{R} = 35^\circ$, $\text{P}\hat{\text{R}}\text{Q} = 25^\circ$.
(Note: in the original question, the first part was to find PR = 5.96)

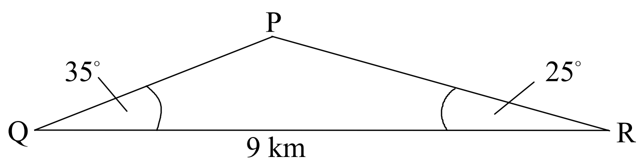
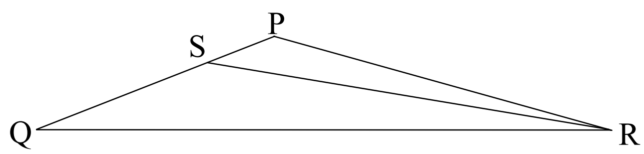


diagram not to scale

(a) Tom sets out to walk from Q to P at a steady speed of 8 km h^{-1}. At the same time, Alan sets out to jog from R to P at a steady speed of $a\text{ km h}^{-1}$. They reach P at the same time. Calculate the value of a. ***[7 marks]***

(b) The point S is on [PQ], such that $\text{RS} = 2\text{QS}$, as shown in the diagram.



Find the length QS. ***[6 marks]***

MARKSCHEME

(a) **EITHER**

Sine rule to find PQ

$\text{PQ} = \dfrac{9\sin 25}{\sin 120}$ ***(M1)(A1)***

$\text{PQ} = 4.39 \text{ km}$ ***(A1)***

OR

Cosine rule: $\text{PQ}^2 = 5.96^2 + 9^2 - (2)(5.96)(9)\cos 25$ ***(M1)(A1)***

$= 19.29$

$\text{PQ} = 4.39 \text{ km}$ ***(A1)***

THEN

Time for Tom $= \dfrac{4.39}{8}$ ***(A1)***

Time for Alan $= \dfrac{5.96}{a}$ ***(A1)***

Then $\dfrac{4.39}{8} = \dfrac{5.96}{a}$ ***(M1)***

$a = 10.9$ ***(A1)*** ***(N5)***

[7 marks]

Note that the **THEN** part follows both **EITHER** and **OR** solutions, and this is shown by the alignment.

(b) **METHOD 1**

$\text{RS}^2 = 4\text{QS}^2$ ***(A1)***

$4\text{QS}^2 = \text{QS}^2 + 81 - 18 \times \text{QS} \times \cos 35$ ***(M1)(A1)***

$3\text{QS}^2 + 14.74\text{QS} - 81 = 0$ (or $3x^2 + 14.74x - 81 = 0$) ***(A1)***

$\Rightarrow \text{QS} = -8.20$ or $\text{QS} = 3.29$ ***(A1)***

therefore $\text{QS} = 3.29$ ***(A1)***

METHOD 2

$\dfrac{\text{QS}}{\sin \text{S}\hat{\text{R}}\text{Q}} = \dfrac{2\text{QS}}{\sin 35}$ ***(M1)***

$\Rightarrow \sin \text{S}\hat{\text{R}}\text{Q} = \dfrac{1}{2}\sin 35$ ***(A1)***

$\text{S}\hat{\text{R}}\text{Q} = 16.7^\circ$ ***(A1)***

Therefore, $\text{Q}\hat{\text{S}}\text{R} = 180 - (35 + 16.7) = 128.3^\circ$ ***(A1)***

$\dfrac{9}{\sin 128.3} = \dfrac{\text{QS}}{\sin 16.7} \left(= \dfrac{\text{SR}}{\sin 35} \right)$ ***(M1)***

$\text{QS} = \dfrac{9\sin 16.7}{\sin 128.3} \left(= \dfrac{9\sin 35}{2\sin 128.3} \right) = 3.29$ ***(A1)*** ***(N2)***

If candidates have shown no working, award ***(N5)*** for the correct answer 10.9 in part (a), and ***(N2)*** for the correct answer 3.29 in part (b). ***[6 marks]***

3. Follow through

Question

Calculate the acute angle between the lines with equations

$$\boldsymbol{r} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + s\begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{r} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Markscheme

Angle between lines = angle between direction vectors. (May be implied) ***(A1)***

Direction vectors are $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. (May be implied) ***(A1)***

$\begin{pmatrix} 4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left|\begin{pmatrix} 4 \\ 3 \end{pmatrix}\right|\left|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right| \cos\theta$ ***(M1)***

$4 \times 1 + 3 \times (-1) = \sqrt{(4^2 + 3^2)}\sqrt{(1^2 + (-1)^2)}\cos\theta$ ***(A1)***

$\cos\theta = \frac{1}{5\sqrt{2}}$ (= 0.1414….) ***(A1)***

$\theta = 81.9^\circ$ (1.43 radians) ***(A1)*** ***(N3)***

Examples of solutions and marking

	Solutions	Marks allocated	
1.	$\begin{pmatrix} 4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left|\begin{pmatrix} 4 \\ 3 \end{pmatrix}\right|\left|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right| \cos\theta$	***(A1)(A1)*** implied ***(M1)***	
	$\cos\theta = \frac{7}{5\sqrt{2}}$	***(A0)(A1)***	
	$\theta = 8.13^\circ$	***(A1)*****ft**	Total 5 marks
2.	$\cos\theta = \frac{\begin{pmatrix} 4 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \end{pmatrix}}{\sqrt{17}\sqrt{20}}$	***(A0)(A0)*** wrong vectors implied ***(M1)*** for correct method, ***(A1)*****ft**	
	$= 0.2169$	***(A1)*****ft**	
	$\theta = 77.5^\circ$	***(A1)*****ft**	Total 4 marks
3.	$\theta = 81.9^\circ$	***(N3)***	Total 3 marks

Note that this candidate has obtained the correct answer, but not shown any working. The way the markscheme is written means that the first 2 marks may be implied by subsequent correct working, but the other marks are only awarded if the relevant working is seen. Thus award the first 2 implied marks, plus the final mark for the correct answer.

END OF EXAMPLES

1. (a) $\dfrac{x^2}{(1+x)(1+x^2)} \equiv \dfrac{a}{(1+x)} + \dfrac{bx+c}{(1+x^2)}$

$x^2 \equiv a(1+x^2)+(bx+c)(1+x)$ ***(M1)(A1)***

$1 = a+b$, $0 = a+c$, $0 = b+c$

Solving gives $1 = 2a$

$a = \dfrac{1}{2} \Rightarrow b = \dfrac{1}{2}$, $c = -\dfrac{1}{2}$. ***(A1)(A1)(A1)*** ***(N2)***

[5 marks]

(b) (i) $I = \dfrac{1}{2}\displaystyle\int \dfrac{1}{(1+x)} + \dfrac{x-1}{(1+x^2)}\,\mathrm{d}x$

$= \dfrac{1}{2}\displaystyle\int \dfrac{1}{(1+x)}\,\mathrm{d}x + \dfrac{1}{4}\int \dfrac{2x}{(1+x^2)}\,\mathrm{d}x - \dfrac{1}{2}\int \dfrac{\mathrm{d}x}{(1+x^2)}$ ***(M1)***

$= \dfrac{1}{2}\ln\left|1+x\right| + \dfrac{1}{4}\ln\left|1+x^2\right| - \dfrac{1}{2}\arctan x + k$ ***(A1)(A1)(A1)***

Note: Do not penalize the absence of *k*, or the absolute value signs.

(ii) $\dfrac{\pi}{4} = \dfrac{1}{2}\ln 2 + \dfrac{1}{4}\ln 2 - \dfrac{\pi}{8} + k$ ***(M1)(A1)***

$\dfrac{3\pi}{8} = \dfrac{3}{4}\ln 2 + k$

$\dfrac{3\pi}{8} - \dfrac{3}{4}\ln 2 = k$ (accept $p = \dfrac{3\pi}{8}$, $q = -\dfrac{3}{4}$, $r = 2$) ***(A1)*** ***(N1)***

Note: I is not unique. Accept equivalent expressions which may lead to different values of p, q, r.

[7 marks]

Total [12 marks]

2. (i) (a) $\det \boldsymbol{M} = \begin{vmatrix} -1 & -k & 3 \\ 4 & 5 & 1 \\ 1 & -1 & k \end{vmatrix} = -1(5k+1) + k(4k-1) + 3(-4-5)$ ***(M1)***

$= -5k - 1 + 4k^2 - k - 27$

$= 4k^2 - 6k - 28$ ***(A1)*** ***(N1)***

[2 marks]

(b) For there not to be a unique solution

$4k^2 - 6k - 28 = 0$ ***(M1)***

$(2k-7)(k+2) = 0$

$k = \frac{7}{2}, -2$ ***(A1)(A1)*** ***(N2)***

[3 marks]

(ii) (a) A vector in the plane is $\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$ ***(M1)(A1)*** ***(N1)***

Normal vector to plane is $\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \\ -6 \end{pmatrix}$ ***(M1)(A1)*** ***(N1)***

Equation of plane is $\boldsymbol{r} \bullet \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix}$

$\boldsymbol{r} \bullet \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix} = 6 - 4 - 9$ ***(M1)(A1)***

$\boldsymbol{r} \bullet \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix} = -7$ ***(A1)***

$\Rightarrow 6x + 2y - 3z = -7$ ***(AG)*** ***(N0)***

[7 marks]

continued ...

Question 2 (ii) continued

(b) **METHOD 1**

Any point P on normal from origin O to plane is $(6k, 2k, -3k)$ ***(M1)***

Distance $\text{OP} = \left| k\sqrt{6^2 + 2^2 + (-3)^2} \right| = |7k|$ ***(A1)***

P lies on plane

$$6(6k) + 2(2k) - 3(-3k) = -7$$
$$36k + 4k + 9k = -7$$
$$k = -\frac{1}{7}$$ ***(A1)***

Distance $= \left| 7 \times -\frac{1}{7} \right| = 1$ ***(A1)*** ***(N2)***

METHOD 2

Using distance $= \left| \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$ ***(M1)***

(x_0, y_0, z_0) is (0, 0, 0)

distance $= \frac{|-7|}{\sqrt{6^2 + 2^2 + (-3)^2}}$ ***(A1)(A1)***

Note: Award ***(A1)*** for the numerator, ***(A1)*** for the denominator.

distance $= \frac{7}{\sqrt{49}} = 1$ ***(A1)*** ***(N2)***

[4 marks]

Total [16 marks]

3. (i) (a) $P(3n)=\frac{1}{6^3}=\frac{1}{216}$; $P(2n)=3\times\frac{5}{216}=\frac{15}{216}$; $P(-n)=\left(\frac{5}{6}\right)^3=\frac{125}{216}$ *(M1)*

Profit	$-n$	n	$2n$	$3n$
Probability	$\mathbf{\frac{125}{216}}$	$\frac{75}{216}$	$\mathbf{\frac{15}{216}}$	$\mathbf{\frac{1}{216}}$

(A1)(A1)(A1) *(N3)*

[4 marks]

(b) $E(X)=(-n)\times\frac{125}{216}+(n)\times\frac{75}{216}+(2n)\times\frac{15}{216}+(3n)\times\frac{1}{216}$ *(M1)(A1)*

$=-\frac{17n}{216}$ *(AG)* *(N0)*

[2 marks]

(c) $-\frac{17n}{216}=-0.34$ *(M1)*

$n=4.32$ (accept \$ 4.32) *(A1)* *(N1)*

[2 marks]

(ii) (a) Let $P(n)$ be the proposition $\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+1)}=\frac{n}{(2n+1)}$

$P(1):\sum_{1}^{1}\frac{1}{(2r-1)(2r+1)}=\frac{1}{3}=\frac{1}{2(1)+1}$ so $P(1)$ is true *(M1)*

Assume that $P(k)$ is true

$P(k+1):\sum_{1}^{k+1}\frac{1}{(2r-1)(2r+1)}=\frac{k}{(2k+1)}+\frac{1}{(2k+1)(2k+3)}$ *(M1)(A1)*

$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$ *(A1)*

$=\frac{(k+1)(2k+1)}{(2k+1)(2k+3)}$

$=\frac{(k+1)}{(2k+1)+1}$ *(A1)*

Therefore $P(1)$ is true and $P(k)\Rightarrow P(k+1)$ so $P(n)$ is true $\forall n\in\mathbb{Z}^+$. *(R1)*

[6 marks]

(b) Checking that $\frac{1}{3}+\frac{1}{15}+\frac{1}{35}$ is the same as $\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+1)}$ *(M1)*

(e.g. substitute $r=1,2$)

Sum is therefore sum of $(n+1)$ terms *(M1)*

i.e. $\frac{(n+1)}{2(n+1)+1}$ *(A1)*

$=\frac{n+1}{2n+3}$ *(AG)*

[3 marks]

Total [17 marks]

4. (a) r^{th} term $= \begin{pmatrix} n \\ n-r \end{pmatrix} x^r h^{n-r} \left(= \dfrac{n!}{r!(n-r)!} x^r h^{n-r} \right)$ ***(A1)***

[1 mark]

(b) $\dfrac{\mathrm{d}(x^n)}{\mathrm{d}x} = \lim\limits_{h \to 0} \left(\dfrac{(x+h)^n - x^n}{h} \right)$ ***(M1)***

$= \lim\limits_{h \to 0} \left(\dfrac{x^n + \begin{pmatrix} n \\ 1 \end{pmatrix} x^{n-1} h + \begin{pmatrix} n \\ 2 \end{pmatrix} x^{n-2} h^2 + \ldots + h^n - x^n}{h} \right)$ ***(A1)***

$= \lim\limits_{h \to 0} \left(\dfrac{x^n + nx^{n-1} h + \dfrac{n(n-1)}{2} x^{n-2} h^2 + \ldots + h^n - x^n}{h} \right)$ ***(A1)***

$= \lim\limits_{h \to 0} \left(nx^{n-1} + \dfrac{n(n-1)}{2} x^{n-2} h + \ldots + h^{n-1} \right)$ ***(A1)***

Note: Accept first, second and last terms in the 3 lines above.

$= nx^{n-1}$ ***(A1)***

[5 marks]

(c) $x^n \times x^{-n} = 1$

$x^n \dfrac{\mathrm{d}(x^{-n})}{\mathrm{d}x} + x^{-n} \dfrac{\mathrm{d}(x^n)}{\mathrm{d}x} = 0$ ***(M1)***

$x^n \dfrac{\mathrm{d}(x^{-n})}{\mathrm{d}x} + x^{-n} \times nx^{n-1} = 0$ ***(A1)***

$x^n \dfrac{\mathrm{d}(x^{-n})}{\mathrm{d}x} + nx^{-1} = 0$ ***(A1)***

$\dfrac{\mathrm{d}(x^{-n})}{\mathrm{d}x} = \dfrac{-nx^{-1}}{x^n} \left(= -nx^{-(1+n)} \right)$ ***(A1)***

[4 marks]

Total [10 marks]

5. (i) (a) $z_1 = 2 + \mathrm{i}$ and $z_2 = 3 + \mathrm{i}$

$z_1 z_2 = (2 + \mathrm{i})(3 + \mathrm{i}) = 5 + 5\mathrm{i}$ ***(A1)***

[1 mark]

(b) (i) $|z_2| = \sqrt{10}$, $\arg z_2 = \arctan\frac{1}{3}$, $|z_1 z_2| = \sqrt{50}$, $\arg z_1 z_2 = \arctan 1$ ***(M1)***

$z_2 = \left(\sqrt{10}, \arctan\frac{1}{3}\right)$, $z_1 z_2 = \left(\sqrt{50}, \arctan 1\right)$ ***(A1)(A1)*** ***(N3)***

(ii) Also $\arg z_1 z_2 = \arg z_1 + \arg z_2$ ***(M1)***

$\arctan 1 = \arctan\frac{1}{2} + \arctan\frac{1}{3}$ ***(A1)***

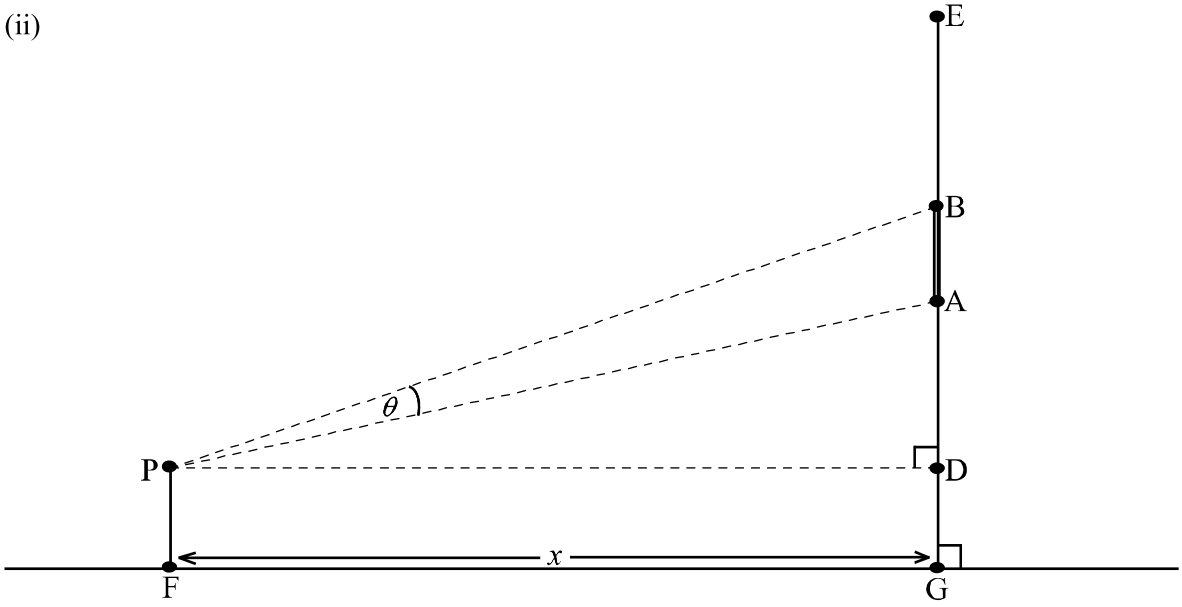
$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}$ ***(AG)*** ***(N0)***

[5 marks]

continued ...

Question 5 continued

(ii)



Let $B\hat{P}D = \alpha$ and $A\hat{P}D = \beta$ then $\theta = \alpha - \beta$.

$$\tan\theta = \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$ ***(M1)***

$$= \frac{\frac{b}{x} - \frac{a}{x}}{1 + \frac{ab}{x^2}} = \frac{(b-a)x}{x^2 + ab}$$ ***(A1)***

$$\frac{\mathrm{d}(\tan\theta)}{\mathrm{d}x} = \frac{(x^2+ab)(b-a) - (b-a)2x^2}{(x^2+ab)^2}$$ ***(M1)***

$$= \frac{(b-a)(ab-x^2)}{(x^2+ab)^2}$$ ***(A1)***

at maximum $(ab - x^2) = 0$, $b \neq a$

$$x = \sqrt{ab}$$ ***(A1)***

$$\frac{\mathrm{d}^2(\tan\theta)}{\mathrm{d}x^2} = (b-a)\left[\frac{(x^2+ab)^2(-2x) - 4x(ab-x^2)(x^2+ab)}{(x^2+ab)^4}\right]$$ ***(M1)***

$$= \frac{(b-a)}{\left(x^2+ab\right)^3}\left[-2x^3 - 2xab - 4xab + 4x^3\right]$$

$$= \frac{(b-a)(2x^3 - 6xab)}{\left(x^2+ab\right)^3}$$ ***(A1)***

at $x = \sqrt{ab}$, $\frac{\mathrm{d}^2(\tan\theta)}{\mathrm{d}x^2} = \frac{(b-a)\left(-4ab\sqrt{ab}\right)}{8a^3b^3}$

$$= \frac{-(b-a)\sqrt{ab}}{2a^2b^2}$$ ***(A1)***

since $\frac{\mathrm{d}^2(\tan\theta)}{\mathrm{d}x^2} < 0$ at $x = \sqrt{ab}$ this value is a maximum. ***(R1)***

[9 marks]

Total [15 marks]

6. (i) (a) $E(2X) = 2E(X) = 2(5) = 10$ ***(A1)***

(b) $\text{Var}(2X) = 4\text{Var}(X) = 12$ ***(A1)***

(c) $E(3X - 2Y) = 3E(X) - 2E(Y) = 3(5) - 2(4) = 7$ ***(A1)***

(d) $\text{Var}(3X - 2Y) = 9\ \text{Var}(X) + 4\ \text{Var}(Y) = 9(3) + 4(2) = 35$ ***(A1)***

[4 marks]

(ii) (a) **METHOD 1**

Sample 1 : Mean $= 9.315 = 9.32$ (3 s.f.) ***(A1)***

Variance $= 0.0171$ (3 s.f.) ***(A1)***

Sample 2 : Mean $= \dfrac{669.6}{72} = 9.3$ ***(A1)***

Variance $= \dfrac{6228}{72} - (9.3)^2$ ***(A1)***

$= 0.01$

Hence pooled estimate for population mean $= \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

$= \dfrac{85(9.315) + 72(9.3)}{85 + 72}$

$= 9.31$ (3 s.f.) ***(A1)***

Hence pooled estimate for population variance $= \dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$ ***(M1)***

$= \dfrac{85(0.0171) + 72(0.01)}{155}$ ***(A1)***

$= 0.0140$ (3 s.f.)

[7 marks]

METHOD 2

Since the samples are drawn from the same population it is also possible to combine the two samples into one for an estimate of population mean and variance.

$\sum x_1 = 791.8$ and $\sum x_2 = 669.6$

$\Rightarrow \sum x = 1461.4$ ***(A1)***

$\Rightarrow \bar{x} = \dfrac{1461.4}{157} = 9.31$ ***(A1)***

$\sum x_1^2 = 7377.3$ and $\sum x_2^2 = 6228$

$\Rightarrow \sum x^2 = 13\,605.3$ ***(A1)***

Now $s_n{}^2 = \dfrac{\sum x^2}{n} - \left(\dfrac{\sum x}{n}\right)^2$

$\Rightarrow s_n{}^2 = \dfrac{13\,605.3}{157} - (9.3083)^2 = 0.01388...$ ***(M1)(A1)***

$\Rightarrow s_{n-1}{}^2 = \dfrac{n}{n-1} s_n{}^2 = 0.01396... = 0.0140$ (3 s.f.) ***(M1)(A1)***

[7 marks]

continued...

Question 6 (ii) continued

(b) Since population variance unknown confidence interval given by $\bar{x} \pm t \frac{s_{n-1}}{\sqrt{n}}$ ***(R1)***

Degrees of freedom are 155 (Method 1); 156 (Method 2)

EITHER

$t = 1.975$ ***(A1)***

CI is $9.31 \pm 1.975\sqrt{\frac{0.01396}{157}}$ ***(A1)***

$=]9.29, 9.33[$ ***(A1)***

OR

Since *n* large, use $z = 1.96$ ***(A1)***

CI is $9.31 \pm 1.96\sqrt{\frac{0.01396}{157}}$ ***(A1)***

$=]9.29, 9.33[$ ***(A1)***

[4 marks]

(iii) (a) $X \sim \mathrm{N}(\mu, \sigma^2)$

EITHER

The sample mean is **normally** distributed ***(R1)***

with mean μ **and** variance $\frac{\sigma^2}{n}$ ***(R1)***

OR

$\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ ***(R1)(R1)***

[2 marks]

(b) H_0 : Mean, $\mu = 1.005$, $\mathrm{H}_2 : \mu \neq 1.005$ ***(A1)***

A two-tail *z*-test is appropriate since σ is given ***(R1)***

EITHER

Sample mean is 1.003 ***(A1)***

$z = \frac{|\bar{x} - \mu|}{\frac{\sigma}{\sqrt{n}}} = \frac{|1.003 - 1.005|}{\frac{0.0028}{\sqrt{8}}}$ ***(M1)***

$= 2.02$ ***(A1)***

Critical *z* value for 1 % test is 2.58 ***(A1)***

Result is not significant, mean is 1.005. ***(A1)***

OR

using gdc $z = -2.02$ ***(A2)***

$p = 0.0434$ ***(A2)***

Result is not significant. Accept H_0 , mean is 1.005. ***(A1)***

[7 marks]

continued…

Question 6 continued

(iv)

Score	1	2	3	4	5	6
Frequency	45	57	51	56	47	44

H_0 : Die is fair.
H_1 : Die is not fair. ***(A1)***

Since 300 throws expect 50 outcomes of each score ***(A1)***

Observed	45	57	51	56	47	44
Expected	50	50	50	50	50	50

$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e}$ ***(M1)***

$= 3.12$ ***(A1)***

From table χ^2 (critical value at 5 % level) with (degrees of freedom = 5) is 11.07 ***(A1)***

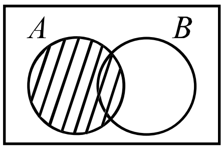
Since $\chi^2_{calc} < 11.07$

Result is not significant, die is fair. ***(R1)***

[6 marks]

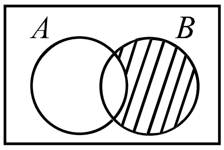
Total [30 marks]

7. (i) (a) $A \cup (B \cap A')'$



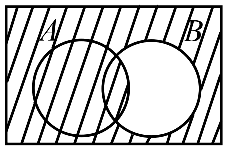
A

and

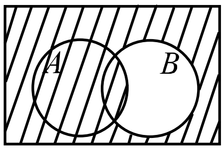


$B \cap A'$

$\Rightarrow$



$A \cup (B \cap A')'$



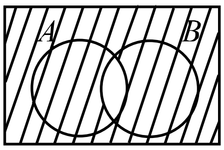
$A \cup B'$

(M1)(A1)

Hence $A \cup (B \cap A')' = A \cup B'$ ***(AG)***

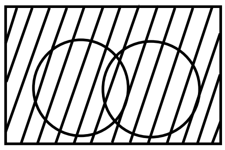
[2 marks]

(b) $\left((A \cap B)' \cup B\right)' = \varnothing$



$(A \cap B)'$

$\Rightarrow$



$(A \cap B)' \cup B$

(A1)

everything shaded ***(R1)***

$\Rightarrow \left((A \cap B)' \cup B\right)' = \varnothing$ ***(AG)***

[2 marks]

(ii) (a) $(M, +)$ is not a group since

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2x \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2x \\ 0 & 2 \end{pmatrix} \notin M$$

We do not have closure. ***(R1)***

Note: Any counter example will do, x term not needed.

[1 mark]

continued…

Question 7 (ii) continued

(b) Under matrix product

$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} \in M \Rightarrow$ closure. ***(A1)***

$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y+x \\ 0 & 1 \end{pmatrix}$ hence operation is commutative ***(A1)***

There is an identity element $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M$ ***(A1)***

Inverses exist since $\begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} \neq 0$ and $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \in M$ ***(M1)(A1)***

Hence M forms an abelian group. ***(AG)***

[5 marks]

(iii) (a)

	a	b	c	d
a	b	$\boldsymbol{c}$	$\boldsymbol{d}$	a
b	$\boldsymbol{c}$	d	$\boldsymbol{a}$	b
c	$\boldsymbol{d}$	$\boldsymbol{a}$	$\boldsymbol{b}$	c
d	a	b	$\boldsymbol{c}$	d

(A4)

Note: Award ***(A3)*** if one error, ***(A2)*** if 2 errors, ***(A1)*** if 3 errors, ***(A0)*** for 4 or more errors in table.

[4 marks]

(b) (i) using inverse elements

$(b\#x)*c*a = d*a$

$\Rightarrow b\#x = a$ ***(A1)***

$\Rightarrow d\#b\#x = d\#a$

$\Rightarrow x = d$ ***(A1)***

(ii) $a*(x\#b)*c*a = b*a$

$\Rightarrow a*(x\#b) = c$ ***(A1)***

$\Rightarrow c*a*(x\#b) = c*c$

$\Rightarrow x\#b = b$ ***(A1)***

$\Rightarrow x\#b\#d = b\#d$

$\Rightarrow x = a$ ***(A1)***

[5 marks]

continued ...

Question 7 continued

(iv) (a) $(a, b)R(p, q) \Rightarrow \max(|a|, |b|) = \max(|p|, |q|)$ ***(M1)***

$\max(|p|, |q|) = \max(|a|, |b|) \Rightarrow (p, q)R(a, b)$

$\Rightarrow R$ is symmetric ***(A1)***

$(a, b)R(a, b) \Rightarrow \max(|a|, |b|) = \max(|a|, |b|)$ ***(M1)***

R is reflexive ***(A1)***

$(a, b)R(x, y)$ and $(x, y)R(p, q) \Rightarrow (a, b)R(p, q)$

since $\max(|a|, |b|) = \max(|x|, |y|)$ and $\max(|x|, |y|) = \max(|p|, |q|)$ ***(M1)***

$\Rightarrow \max(|a|, |b|) = \max(|p|, |q|)$

R is transitive. ***(A1)***

$\Rightarrow R$ is an equivalence relation. ***(AG)***

[6 marks]

(b) (i) If $\max(|x|, |y|) = c$

Then $|x| = c$ and $|y| \le c$

$\Rightarrow x = \pm c$ and $-c \le y \le c$ ***(M1)(A1)***

or $|y| = c$ and $|x| \le c$ ***(M1)***

$\Rightarrow y = \pm c$ and $-c \le x \le c$ ***(A1)***

(ii) i.e. Concentric squares with a centre at (0, 0) ***(A1)***

[5 marks]

Total [30 marks]

8. (i) $\gcd(64, 33) = \gcd(33, 64 \bmod 33)$ *(M1)*

$= \gcd(33, 31)$

$= \gcd(31, 33 \bmod 31)$ *(M1)*

$= \gcd(31, 2)$

$= \gcd(2, 31 \bmod 2)$

$= \gcd(2, 1)$

$= 1$ *(A1)*

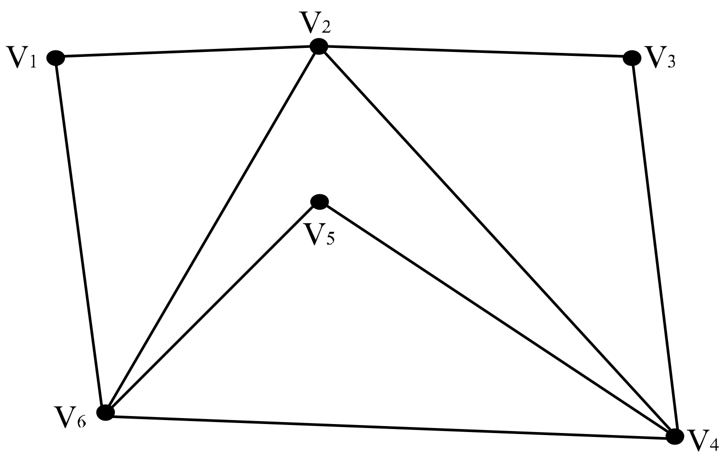
hence 64 and 33 are relatively prime. *(AG)*

[3 marks]

(ii) $\mathbb{Z}$ is not well ordered because it contains subsets (e.g. $\mathbb{Z}$ itself) which do not have a smallest element. *(A2)*

[2 marks]

(iii)



(a) (i) **EITHER**

Every vertex has even degree ⇒ Eulerian circuit exists. *(A1)*

OR

Circuit containing all edges is

$V_1, V_2, V_3, V_4, V_2, V_6, V_5, V_4, V_6, V_1$. *(A1)*

(ii) A cycle containing all vertices is

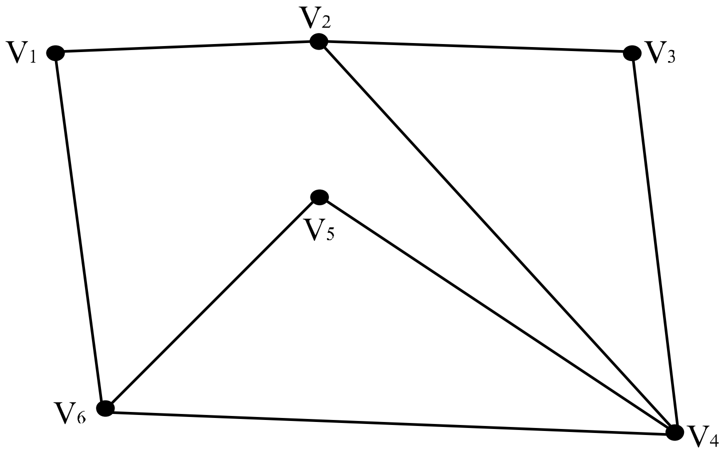
$V_1, V_2, V_3, V_4, V_5, V_6, V_1$. *(A2)*

[3 marks]

continued …

Question 8 (iii) continued

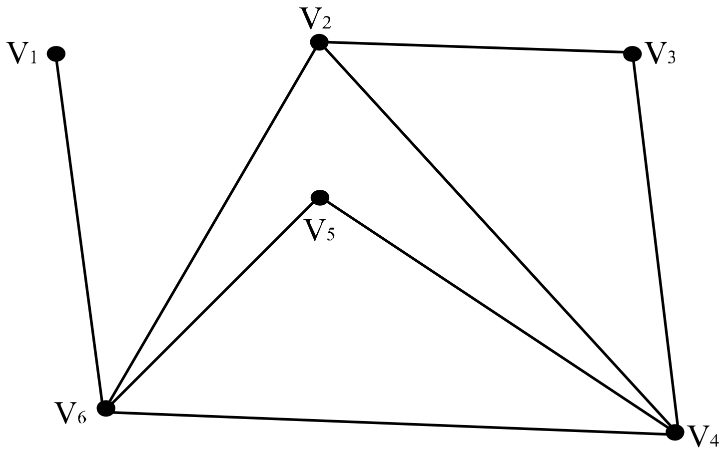
(b) Removing edge V_2 V_6



There is no Eulerian circuit since V_2 and V_6 are now odd degree. ***(M1)(A1)***
There is a Hamiltonian cycle still, same as above. ***(A1)***

[3 marks]

(c) If we now replace edge V_2 V_6 and remove V_1 V_2



(i) an Eulerian trail
$V_2, V_3, V_4, V_2, V_6, V_5, V_4, V_6, V_1$ ***(A2)***

(ii) a Hamiltonian path
$V_2, V_3, V_4, V_5, V_6, V_1$ ***(A2)***

Note: Other solutions are possible.

[4 marks]

continued …

Question 8 continued

(iv) (a) $u_n = u_{n-1} + u_{n-2}$ for $n \geq 2, u_0 = u_1 = 1$

$\Rightarrow$ 1, 1, 2, 3, 5, 8, 13, 21, … ***(A1)***

[1 mark]

(b) $r^2 - r - 1 = 0$ is the characteristic equation. ***(A1)***

$$\Rightarrow r = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\Rightarrow r_1 = \frac{1+\sqrt{5}}{2} \text{ and } r_2 = \frac{1-\sqrt{5}}{2}$$ ***(A1)(A1)***

$$\Rightarrow u_n = A(r_1)^n + B(r_2)^n$$ ***(M1)***

Now $n = 0 \Rightarrow u_0 = 1 \Rightarrow 1 = A + B$ (1)

and $n = 1 \Rightarrow u_1 = 1 \Rightarrow 1 = Ar_1 + Br_2$ (2) ***(A1)***

Solving simultaneously for *A* and *B*

from (1) $B = 1 - A$

Subsititute in (2) $\Rightarrow 1 = Ar_1 + (1-A)r_2 \Rightarrow 1 = A(r_1 - r_2) + r_2$

$$A = \frac{1 - r_2}{r_1 - r_2} = \frac{1 - \left(\frac{1-\sqrt{5}}{2}\right)}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)}{\sqrt{5}}$$ ***(A1)***

$$B = 1 - \frac{\left(\frac{1+\sqrt{5}}{2}\right)}{\sqrt{5}} = \frac{\sqrt{5} - \left(\frac{1+\sqrt{5}}{2}\right)}{\sqrt{5}} = \frac{\left(\frac{-1+\sqrt{5}}{2}\right)}{\sqrt{5}}$$ ***(A1)***

$$u_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{-1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$ ***(A1)***

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$ ***(AG)***

[8 marks]

continued ...

Question 8 (iv) continued

(c) (i) Since $\left|\frac{-1+\sqrt{5}}{2}\right| < 0.62$ ***(A1)***

$\frac{1}{\sqrt{5}}\left|\frac{-1+\sqrt{5}}{2}\right|^{n+1} < 0.5$ for $n \geq 0$ ***(A1)***

$\Rightarrow$ the terms are getting smaller and smaller as n increases. ***(A1)***

$\Rightarrow u_n$ is given by the closest integer to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$ ***(AG)*** ***(N0)***

(ii) $u_n = 102\,334\,155$

$\Rightarrow 102\,334\,155 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$

$\Rightarrow (n+1)\log\left(\frac{1+\sqrt{5}}{2}\right) = \log\left[(102\,334\,155)(\sqrt{5})\right]$ ***(M1)(A1)***

$\Rightarrow n+1 = 40$

$\Rightarrow n = 39$ (So $102\,334\,155$ is the 40^{th} term of this sequence) ***(A1)***

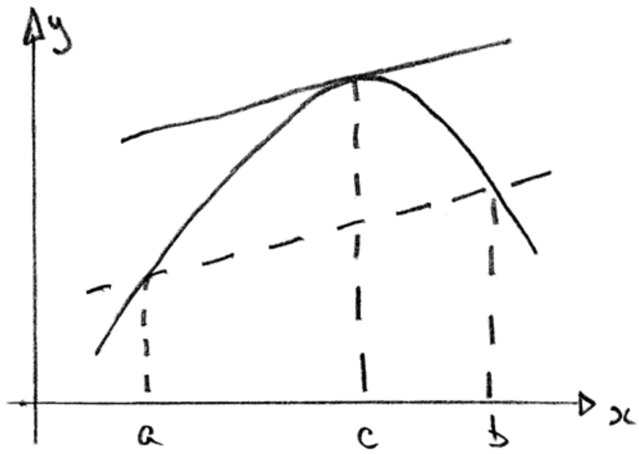
[6 marks]

Total [30 marks]

9. (i) (a) If a function f is continuous on a closed interval $[a,b]$ and is differentiable on the open interval $]a,b[$ then there exists a number c in $]a,b[$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a} \quad \textbf{OR} \quad f(b)-f(a)=f'(c)(b-a)$$ ***(A1)***

This can be illustrated with the following sketch.



(A1)

[2 marks]

(b) If $f'(x)=0 \Rightarrow f(p)-f(q)=0$ for all values of p and q in interval $[a, b]$ ***(M1)***

$\Rightarrow f(p)=f(q)$ ***(A1)***

and f is constant on the interval ***(AG)***

[2 marks]

(ii) (a) $\int_0^2 3x^5\,\mathrm{d}x=\left[\frac{3x^6}{6}\right]_0^2$

$=32$ ***(A1)***

[1 mark]

(b) $\int_0^2 3x^5\mathrm{d}x \approx \frac{h}{3}\left[\,y_0+4y_1+2y_2+4y_3+y_4\,\right]$ ***(M1)***

where $h=\frac{2-0}{4}=\frac{1}{2}$

Using the following table of x, y values

x	y
0	0
0.5	0.09375
1	3
1.5	22.78125
2	96

(A1)(A1)

Note: Award ***(A1)*** for x-values, ***(A1)*** for y-values.

$\int_0^2 3x^5\mathrm{d}x \approx \frac{1}{6}\left[\,0+4(0.09375)+2(3)+4(22.78125)+96\,\right]$

$=32.25$ (accept 32.3) ***(A1)***

[4 marks]

continued ...

Question 9 (ii) continued

(c) Error $= 0.25$ ***(A1)***

[1 mark]

(d) Error $\le \dfrac{(b-a)h^4}{180}\left| f^{(4)}(c) \right|$

Now $f(x) = 3x^5$

$f'(x) = 15x^4$

$f''(x) = 60x^3$

$f'''(x) = 180x^2$

$f^4(x) = 360x$ ***(A1)***

So over $[0, 2]$ max $f^{(4)}(x) = 720$ ***(A1)***

$\Rightarrow \dfrac{2}{180}\left(\dfrac{2}{n}\right)^4 (720) < 0.0001$ ***(M1)***

$\Rightarrow \dfrac{16}{n^4} < 1.25 \times 10^{-5}$

$\Rightarrow n^4 > 1\,280\,000$

$\Rightarrow n > 33.6$ ***(M1)***

$\Rightarrow$ error < 0.0001

34 intervals needed ***(A1)***

[5 marks]

(iii) (a) (i) $\sum (-1)^{n-1} \dfrac{1}{(2n-1)!}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{(2n-1)!} = 0$ ***(A1)***

Now $\dfrac{1}{(2n-1)!}$ is decreasing as n increases

$\Rightarrow |a_n| > |a_{n+1}|$ for $n \ge 1$ ***(A1)***

So by alternating series test ***(M1)***

$\sum (-1)^{n-1} \dfrac{1}{(2n-1)!}$ is convergent. (accept ratio test) ***(AG)***

(ii) $S_4 = 1 - \dfrac{1}{3!} + \dfrac{1}{5!} - \dfrac{1}{7!}$ ***(M1)***

$= 1 - \dfrac{1}{6} + \dfrac{1}{120} - \dfrac{1}{5040}$ ***(A1)***

$= 0.841468$ (6 d.p.)

(iii) Error in n^{th} partial sum is less than a_{n+1}

$\Rightarrow S_4$ Error $< a_5$

$\Rightarrow$ Error $< \dfrac{1}{9!}$ ***(M1)***

$\Rightarrow$ Error < 0.00000276 ***(A1)***

[7 marks]

continued ...

Question 9 (iii) continued

(b) (i) $f(x) = \sin x \quad f(0) = 0$

$f'(x) = \cos x \quad f'(0) = 1$

$f''(x) = -\sin x \quad f''(0) = 0$

$f'''(x) = -\cos x \quad f'''(0) = -1$

$f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x \quad f^{(5)}(0) = 1$

$f^{(6)}(x) = -\sin x \quad f^{(6)}(0) = 0$

$f^{(7)}(x) = -\cos x \quad f^{(7)}(0) = -1$

$\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ ***(M1)(A1)***

(ii) n^{th} term given by $(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$ ***(A1)(A1)***

> **Note:** Award ***(A1)*** for $(-1)^{n-1}$, ***(A1)*** for $\frac{x^{2n-1}}{(2n-1)!}$.

(iii) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{x^{2n+1}}{(2n+1)!}\right)\left(\frac{(2n-1)!}{x^{2n-1}}\right)$

$= \lim_{n\to\infty}\frac{x^2}{(2n+1)2n}$

$= 0$ ***(M1)(A1)***

series converges for all *x*. ***(AG)***

(iv) Now $\cos x = \frac{\mathrm{d}(\sin x)}{\mathrm{d}x}$

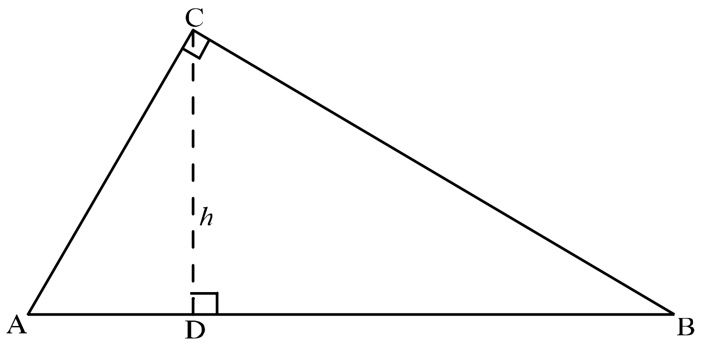
$= \frac{\mathrm{d}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)}{\mathrm{d}x}$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ ***(M1)(A1)***

[8 marks]

Total [30 marks]

10. (i) (a)



$\hat{CAD} = 90^\circ - \hat{ABC}$, $\hat{BCD} = 90^\circ - \hat{ABC}$, or $\hat{ACD} = 90^\circ - \hat{CAB}$, $\hat{DBC} = 90^\circ - \hat{CAB}$

$\hat{CAD} = \hat{BCD}$ or $\hat{ACD} = \hat{DBC}$ or $\hat{ADC} = \hat{BDC}$ $(= 90^\circ)$

Since two angles in Δ ACD are equal to two angles in Δ CDB *(A1)(A1)*

Note: Award ***(A1)*** for noting one correct pair of equal angles and ***(A1)*** for a second pair **and** the statement.

$\Rightarrow \Delta$ ADC is similar to Δ BCD *(AG)*

[2 marks]

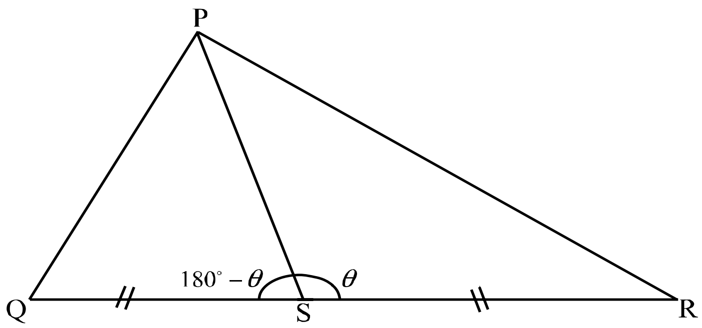
(b) Corresponding sides of Δ s are in equal proportion *(M1)*

$\Rightarrow \frac{CD}{BD} = \frac{AD}{CD}$ *(A1)*

$\Rightarrow CD^2 = AD \times BD$ *(AG)*

[2 marks]

(ii)



Let $\hat{PSR} = \theta \Rightarrow \hat{PSQ} = 180^\circ - \theta$ *(A1)*

Using Δ PRS we obtain $\cos\theta = \frac{SR^2 + PS^2 - PR^2}{2 \times SR \times PS}$ *(M1)(A1)*

Using Δ PQS we obtain $\cos(180^\circ - \theta) = \frac{QS^2 + PS^2 - PQ^2}{2 \times QS \times PS}$ *(A1)*

Now $\cos(180^\circ - \theta) = -\cos\theta$ *(A1)*

$-\frac{QS^2 + PS^2 - PQ^2}{2 \times QS \times PS} = \frac{SR^2 + PS^2 - PR^2}{2 \times QS \times PS}$ *(M1)*

$QS^2 + SR^2 + 2PS^2 = PR^2 + PQ^2$ *(A1)*

Since SR = QS *(A1)*

$PQ^2 + PR^2 = 2(PS^2 + QS^2)$ *(AG)*

[8 marks]

continued ...

Question 10 continued

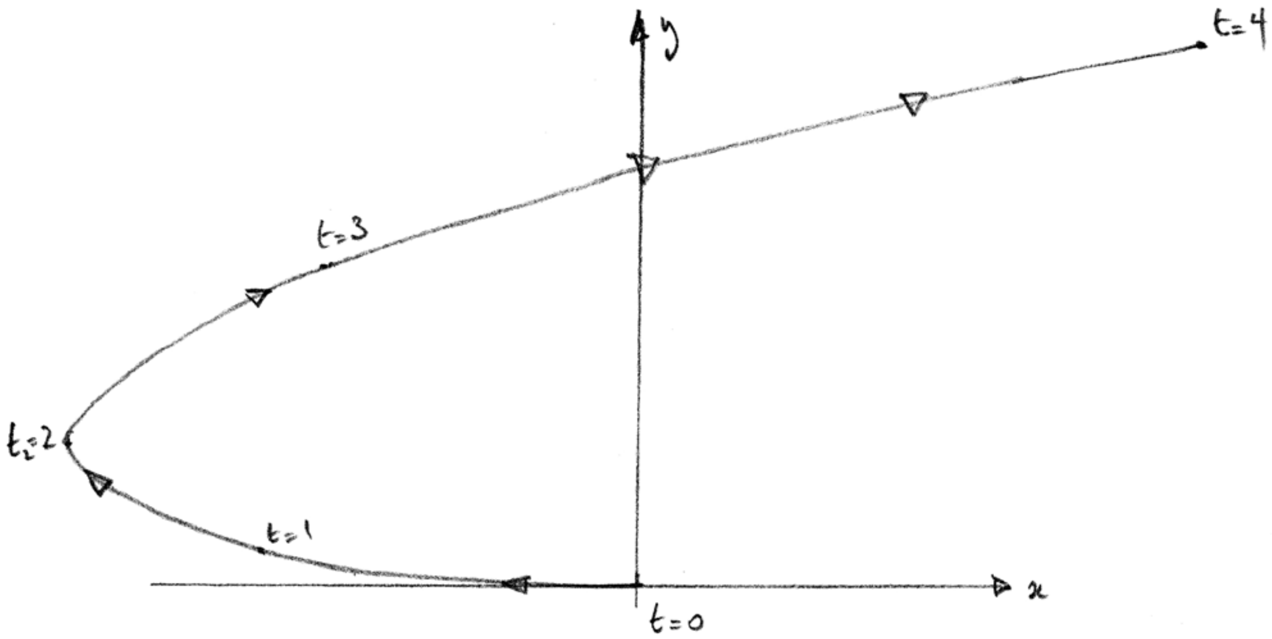
(iii) (a) $x = \frac{1}{2}t^3 - 6t$

$y = \frac{1}{2}t^2$

t	x	y
1	–5.5	0.5
2	–8	2
3	–4.5	4.5
4	8	8

(A1)(A1)

Note: Award ***(A1)*** for x values, ***(A1)*** for y values.



(A1)(A2)

Note: Award ***(A2)*** for sketch, ***(A1)*** for direction of motion.

[5 marks]

continued ...

Question 10 (iii) continued

(b) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ ***(M1)***

$= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t}{\frac{3}{2}t^2 - 6}$ ***(A1)(A1)***

at $t = t_1$ $m = \frac{t_1}{\frac{3}{2}t_1^2 - 6}$, $x = \frac{1}{2}t_1^3 - 6t_1$ and $y = \frac{1}{2}t_1^2$

Hence the equation of the tangent is given by

$y - \frac{1}{2}t_1^2 = \frac{t_1}{\frac{3}{2}t_1^2 - 6}\left(x - \left(\frac{1}{2}t_1^3 - 6t_1\right)\right)$ ***(M1)(A1)***

$\left(y - \frac{1}{2}t_1^2\right)\left(\frac{3}{2}t_1^2 - 6\right) = t_1\left(x - \frac{1}{2}t_1^3 + 6t_1\right)$

$\frac{3}{2}yt_1^2 - 6y - \frac{3}{4}t_1^4 + 3t_1^2 = t_1x - \frac{1}{2}t_1^4 + 6t_1^2$ ***(A1)***

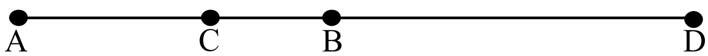
$\frac{3}{2}yt_1^2 - t_1x - 6y = \frac{1}{4}t_1^4 + 3t_1^2$

$6yt_1^2 - 4t_1x - 24y = t_1^4 + 12t_1^2$ ***(A1)***

$-4t_1x + 6y(t_1^2 - 4) = t_1^4 + 12t_1^2$ ***(AG)***

[7 marks]

(iv) The points A, B, C and D are such that $\frac{AC}{CB} = \frac{AD}{BD}$



Let $AB = b$, $AC = c$, $AD = d$

$\frac{c}{b - c} = \frac{d}{d - b}$ ***(M1)(A1)***

$cd - bc = bd - cd$ ***(A1)***

$2cd = bd + bc$ ***(A1)***

$2cd = b(c + d)$ ***(A1)***

$b = \frac{2cd}{c + d}$ ***(A1)***

Hence AB is the harmonic mean of AC and AD. ***(AG)***

[6 marks]

Total [30 marks]