MATHEMATICS
HIGHER LEVEL
PAPER 2

Friday 4 November 2005 (morning)

3 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.

Please start each question on a new page. You are advised to show all working, where possible. Where an answer is wrong, some marks may be given for correct method, provided this is shown by written working. Solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer.

SECTION A

Answer all ***five*** *questions from this section.*

1. *[Maximum mark: 12]*

(a) Given that $\dfrac{x^2}{(1+x)(1+x^2)} \equiv \dfrac{a}{(1+x)} + \dfrac{bx+c}{(1+x^2)}$, calculate the value of a, of b and of c. *[5 marks]*

(b) (i) Hence, find $I = \displaystyle\int \frac{x^2}{(1+x)(1+x^2)}\,\mathrm{d}x$.

(ii) If $I = \dfrac{\pi}{4}$ when $x = 1$, calculate the value of the constant of integration giving your answer in the form $p + q\ln r$ where $p, q, r \in \mathbb{R}$. *[7 marks]*

2. *[Maximum mark: 16]*

(i) (a) Let $\boldsymbol{M} = \begin{pmatrix} -1 & -k & 3 \\ 4 & 5 & 1 \\ 1 & -1 & k \end{pmatrix}$. Find $\det \boldsymbol{M}$. *[2 marks]*

(b) Find the values of k for which the following system of equations does **not** have a unique solution.

$$\begin{aligned} -x - ky + 3z &= -1 \\ 4x + 5y + z &= 2 \\ x - y + kz &= 1 \end{aligned}$$

[3 marks]

(ii) The plane π contains the line $\dfrac{x-1}{2} = \dfrac{y-1}{3} = \dfrac{z-5}{6}$ and the point (1, –2, 3).

(a) Show that the equation of π is $6x + 2y - 3z = -7$. *[7 marks]*

(b) Calculate the distance of the plane π from the origin. *[4 marks]*

3. *[Maximum mark: 17]*

(i) In a game a player pays an entrance fee of \$ n. He then selects one number from 1, 2, 3, 4, 5, 6 and rolls three standard dice.

If his chosen number appears on all three dice he wins four times his entrance fee.

If his number appears on exactly two of the dice he wins three times the entrance fee.

If his number appears on exactly one die he wins twice the entrance fee.

If his number does not appear on any of the dice he wins nothing.

(a) Copy and complete the probability table below. *[4 marks]*

Profit (\$)	$-n$	n	$2n$	$3n$
Probability		$\frac{75}{216}$		

(b) Show that the player's expected profit is \$ $\left(-\frac{17n}{216}\right)$. *[2 marks]*

(c) What should the entrance fee be so that the player's expected loss per game is 34 cents? *[2 marks]*

(ii) (a) Use mathematical induction to prove that

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1},\ n \in \mathbb{Z}^{+}.$$ *[6 marks]*

(b) Hence show that the sum of the first $(n+1)$ terms of the series $\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}+\ldots$ is $\frac{(n+1)}{(2n+3)}$. *[3 marks]*

4. *[Maximum mark: 10]*

(a) Write down the term in x^r in the expansion of $(x+h)^n$, where $0 \le r \le n$, $n \in \mathbb{Z}^+$. *[1 mark]*

(b) Hence differentiate x^n, $n \in \mathbb{Z}^+$, from first principles. *[5 marks]*

(c) Starting from the result $x^n \times x^{-n} = 1$, deduce the derivative of x^{-n}, $n \in \mathbb{Z}^+$. *[4 marks]*

5. *[Maximum mark: 15]*

(i) The complex numbers z_1 and z_2 are $z_1 = 2 + \mathrm{i}$, $z_2 = 3 + \mathrm{i}$.

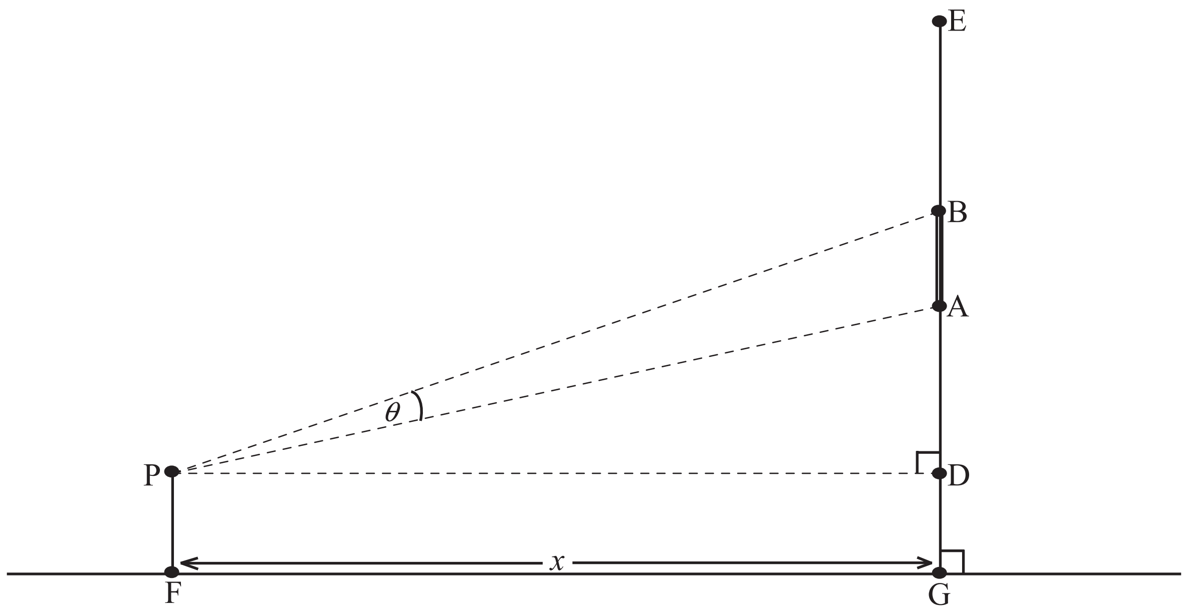
(a) Find $z_1 z_2$, giving your answer in the form $a + \mathrm{i}b$, $a, b \in \mathbb{R}$. *[1 mark]*

(b) The polar form of z_1 may be written as $\left(\sqrt{5}, \arctan\frac{1}{2}\right)$.

(i) Express the polar form of z_2, $z_1 z_2$ in a similar way.

(ii) Hence show that $\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}$. *[5 marks]*

(ii) A man PF is standing on horizontal ground at F at a distance x from the bottom of a vertical wall GE. He looks at the picture AB on the wall. The angle BPA is θ.



Let $\mathrm{DA} = a$, $\mathrm{DB} = b$, where angle PDE is a right angle. Find the value of x for which $\tan\theta$ is a maximum, giving your answer in terms of a and b. Justify that this value of x does give a maximum value of $\tan\theta$. *[9 marks]*

SECTION B

*Answer **one** question from this section.*

Statistics

6. *[Maximum mark: 30]*

(i) Let X and Y be two independent variables with $\mathrm{E}(X) = 5$, $\mathrm{Var}(X) = 3$, $\mathrm{E}(Y) = 4$, $\mathrm{Var}(Y) = 2$. Find

(a) $\mathrm{E}(2X)$;

(b) $\mathrm{Var}(2X)$;

(c) $\mathrm{E}(3X - 2Y)$;

(d) $\mathrm{Var}(3X - 2Y)$. *[4 marks]*

(ii) (a) Two samples are drawn from a normal population that has an unknown mean μ and unknown variance σ^2.

The results of the first sample are given in the following table.

x	9.1	9.2	9.3	9.4	9.5
Frequency	12	16	19	23	15

The second sample of 72 items gave the following results

$$\sum x = 669.6 \text{ and } \sum x^2 = 6228.$$

Use the two samples to calculate an estimate for μ and for σ^2. *[7 marks]*

(b) Based on the combined sample data find a 95 % confidence interval for μ. *[4 marks]*

(This question continues on the following page)

(Question 6 continued)

(iii) (a) A sample of size n is drawn from a population which is normally distributed with mean μ and variance σ^2. Describe fully how the sample mean is distributed. *[2 marks]*

(b) A machine shop manufactures steel rods for use in a car production plant. The lengths in metres for a sample of 8 rods are given below.

0.999, 1.001, 1.005, 1.011, 1.005, 1.001, 0.998, 1.004

Previous observations have shown that the machine settings gave rods with lengths that are normally distributed with standard deviation 0.0028 m.

Stating the type of test used, determine at the 1 % significance level if the mean length of the rods produced is 1.005 m. *[7 marks]*

(iv) A six-sided die is thrown 300 times and the outcomes recorded in the following table.

Score	1	2	3	4	5	6
Frequency	45	57	51	56	47	44

Perform a suitable test at the 5 % significance level to determine if the die is fair. *[6 marks]*

Sets, Relations and Groups

7. *[Maximum mark: 30]*

(i) Use Venn diagrams to show that

(a) $A \cup (B \cap A')' = A \cup B'$; *[2 marks]*

(b) $\left((A \cap B)' \cup B\right)' = \varnothing$. *[2 marks]*

(ii) Let M be the set of all matrices of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ where $x \in \mathbb{R}$.

(a) Show that $(M, +)$ is not a group. *[1 mark]*

(b) Show that M forms an abelian group under matrix multiplication. (You may assume that matrix multiplication is associative). *[5 marks]*

(iii) The set $S = \{a, b, c, d\}$ forms a group under each of two operations # and $*$, as shown in the following group tables.

#	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

$*$	a	b	c	d
a	b			a
b		d		b
c				c
d	a	b		d

(a) Copy and complete the second table. *[4 marks]*

(b) Solve the following equations for x.

(i) $(b \# x) * c = d$.

(ii) $\left(a * (x \# b)\right) * c = b$. *[5 marks]*

(iv) Let $\max\left(|x|, |y|\right)$ be equal to the largest of $|x|$ and $|y|$. Define the relation R on the xy plane by

$$(a, b)\, R\, (p, q) \Leftrightarrow \max\left(|a|, |b|\right) = \max\left(|p|, |q|\right).$$

(a) Show that the relation R is an equivalence relation. *[6 marks]*

(b) (i) Find the equivalence classes.

(ii) Hence describe the equivalence classes. *[5 marks]*

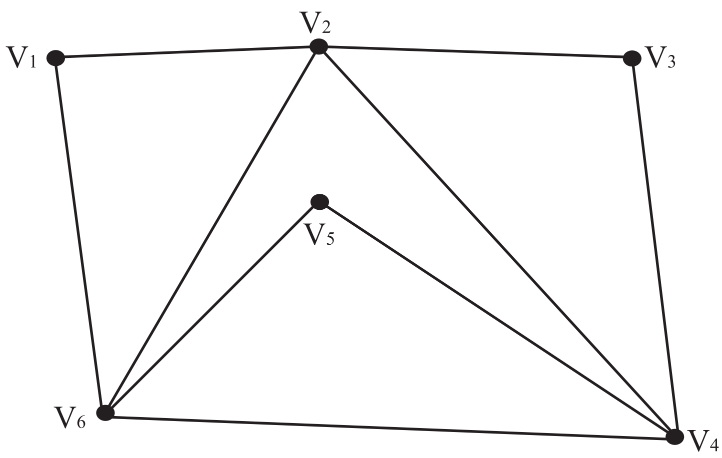
Discrete Mathematics

8. *[Maximum mark: 30]*

(i) Using Euclid's algorithm, show that 64 and 33 are relatively prime. *[3 marks]*

(ii) Explain why $\mathbb{Z}$ is not well-ordered. *[2 marks]*

(iii) Consider the following graph.



(a) Show that this graph has

(i) an Eulerian circuit;

(ii) a Hamiltonian cycle. *[3 marks]*

(b) The edge joining V_2 and V_6 is removed. Does the graph still have an Eulerian circuit and a Hamiltonian cycle? Give reasons for your answers. *[3 marks]*

(c) Replace the edge joining V_2 and V_6, and remove the edge joining V_1 and V_2.

(i) Find an Eulerian trail.

(ii) Find a Hamiltonian path. *[4 marks]*

(This question continues on the following page)

(Question 8 continued)

(iv) The Fibonacci sequence is defined by the recurrence relation

$$u_n = u_{n-1} + u_{n-2}\text{, for } n \ge 2 \text{ and } u_0 = u_1 = 1.$$

(a) Write out the first eight terms of the sequence. *[1 mark]*

(b) Solve the recurrence relation to obtain the formula

$$u_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)\text{, for } n \ge 0.$$ *[8 marks]*

(c) (i) Hence show that u_n is also equal to the closest integer to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$.

(ii) Given that $u_n = 102\,334\,155$, find the value of *n*. *[6 marks]*

Analysis and Approximation

9. *[Maximum mark: 30]*

(i) (a) State the mean value theorem and illustrate it with the aid of a sketch. *[2 marks]*

(b) Use the mean value theorem to prove that if $f'(x) = 0$ for all x in a closed interval then f is constant on that interval. *[2 marks]*

(ii) (a) Find $\int_0^2 3x^5 \, dx$. *[1 mark]*

(b) Use Simpson's Rule with four sub-intervals to approximate $\int_0^2 3x^5 \, dx$. *[4 marks]*

(c) What is the error in this approximation? *[1 mark]*

(d) How many sub-intervals are necessary for the error to be less than 0.0001? *[5 marks]*

(iii) (a) (i) Prove that the alternating series given by $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)!}$ converges.

(ii) Approximate the series by finding the 4th partial sum. Give your answer to **six** decimal places.

(iii) What is the upper bound for the error in this approximation? *[7 marks]*

(b) (i) Find the first four non-zero terms of the Maclaurin series for $\sin x$.

(ii) Deduce the n^{th} term of this series.

(iii) Use the ratio test to show that the series is convergent for all values of x.

(iv) Use your series for $\sin x$ to find the first four non-zero terms of the Maclaurin series for $\cos x$. *[8 marks]*

Euclidean Geometry and Conic Sections

10. *[Maximum mark: 30]*

(i) Consider the triangle ABC, where C is a right angle. Let D be the point on [AB] such that (CD) is perpendicular to (AB).

(a) Show that triangle ADC is similar to triangle CDB. *[2 marks]*

(b) Hence show that $\mathrm{CD}^2 = \mathrm{AD} \times \mathrm{BD}$. *[2 marks]*

(ii) If S is the midpoint of the base [QR] of a triangle PQR, prove Apollonius' theorem

$$\mathrm{PQ}^2 + \mathrm{PR}^2 = 2(\mathrm{PS}^2 + \mathrm{QS}^2).$$ *[8 marks]*

(iii) (a) In the xy plane a particle moves such that $x = \frac{1}{2}t^3 - 6t$ and $y = \frac{1}{2}t^2$, where t is time in seconds. Sketch the path of the particle over the interval $0 \le t \le 4$. Indicate clearly on the sketch the direction of motion of the particle. *[5 marks]*

(b) Show that the equation of the tangent to the curve at $t = t_1$ can be expressed in the form $-4t_1 x + 6y(t_1^2 - 4) = t_1^4 + 12t_1^2$. *[7 marks]*

(iv) The harmonic mean of p and q is given by $\dfrac{2pq}{p+q}$.

Let the points C and D divide the line [AB] such that $\dfrac{\mathrm{AC}}{\mathrm{CB}} = \dfrac{\mathrm{AD}}{\mathrm{BD}}$, as shown in the diagram below.

Show that AB is the harmonic mean of AC and AD. *[6 marks]*