MARKSCHEME

May 2005

MATHEMATICS

Higher Level

Paper 2

Instructions to Examiners

Note: **Where there are two marks (*e.g.* M2, A2) for an answer do not split the marks unless otherwise instructed.**

1 Method of marking

(a) All marking must be done using a **red** pen.

(b) Marks should be noted on candidates' scripts as in the markscheme:

- show the breakdown of individual marks using the abbreviations ***(M1)***, ***(A2)*** *etc*., unless a part is completely correct;
- write down each part mark total, indicated on the markscheme (for example, ***[3 marks]***) – it is suggested that this be written at the end of each part, and underlined;
- write down and circle the total for each question at the end of the question.

2 Abbreviations

The markscheme may make use of the following abbreviations:

(M) Marks awarded for **Method**

(A) Marks awarded for an **Answer** or for **Accuracy**

(N) Marks awarded for correct answers, if **no** working (or no relevant working) shown: they may not necessarily be all the marks for the question. Examiners should only award these marks for correct answers where there is no working.

(R) Marks awarded for clear **Reasoning**

(AG) **Answer Given** in the question and consequently marks are **not** awarded

Note: Unless otherwise stated, it is not possible to award ***(M0)(A1).***

Follow through (ft) marks should be awarded where a correct method has been attempted but error(s) are made in subsequent working which is essentially correct.

- Penalize the error when it first occurs
- Accept the incorrect result as the appropriate quantity in all subsequent working
- If the question becomes much simpler then use discretion to award fewer marks

Examiners should use **(d)** to indicate where discretion has been used. It should only be used for decisions on follow through and alternative methods. It must be accompanied by a brief note to explain the decision made

3 Using the Markscheme

(a) This markscheme presents a particular way in which each question may be worked and how it should be marked. **Alternative methods** have not always been included. Thus, if an answer is wrong then the working must be carefully analysed in order that marks are awarded for a different method in a manner which is consistent with the markscheme. Indicate the awarding of these marks by (**d**).

Where alternative methods for complete questions or parts of questions are included, they are indicated by **METHOD 1**, **METHOD 2**, *etc*. Other alternative part solutions are indicated by **EITHER**….**OR.** It should be noted that ***G*** marks have been removed, and GDC solutions will not be indicated using the **OR** notation as on previous markschemes.

Candidates are expected to show working on this paper, and examiners should not award full marks for just the correct answer. Where it is appropriate to award marks for correct answers with no working (or no relevant working), it will be shown on the markscheme using the ***N*** notation. All examiners will be expected to award marks accordingly in these situations.

(b) Unless the question specifies otherwise, accept **equivalent forms**. For example: $\frac{\sin\theta}{\cos\theta}$ for $\tan\theta$.

On the markscheme, these equivalent numerical or algebraic forms will generally be written in brackets after the required answer. Paper setters will indicate the required answer, by allocating full marks at that point. Further working should be ignored, even if it is incorrect. For example: if candidates are asked to factorize a quadratic expression, and they do so correctly, they are awarded full marks. If they then continue and find the roots of the corresponding equation, do not penalize, even if those roots are incorrect, *i.e.* once the correct answer is seen, ignore further working.

(c) As this is an international examination, all **alternative forms of notation** should be accepted. For example: 1.7 , 1·7, 1,7; different forms of vector notation such as $\vec{u}$, $\bar{u}$, $\underline{u}$; $\tan^{-1}x$ for arctan x.

4 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

There are two types of accuracy error. Candidates should be penalized **once only IN THE PAPER** for an accuracy error **(AP).**

Award the marks as usual then write –1(**AP**) against the answer and also on the **front** cover

Rounding errors: only applies to final answers not to intermediate steps.

Level of accuracy: when this is not specified in the question the general rule *unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures* applies.

- If a final correct answer is incorrectly rounded, apply the **AP**
  OR
- If the level of accuracy is not specified in the question, apply the **AP** for answers not given to 3 significant figures. (Please note that this has changed from 2003).

Note: **If there is no working shown**, and answers are given to the correct two significant figures, apply the **AP**. However, do **not** accept answers to one significant figure without working.

5 Graphic Display Calculators

Many candidates will be obtaining solutions directly from their calculators, often without showing any working. They have been advised that they must use mathematical notation, not calculator commands when explaining what they are doing. Incorrect answers without working will receive no marks. However, if there is written evidence of using a graphic display calculator correctly, method marks may be awarded. Where possible, examples will be provided to guide examiners in awarding these method marks.

Examples

1. Accuracy

A question leads to the answer 4.6789….

- 4.68 is the correct 3 s.f. answer.
- 4.7, 4.679 are to the wrong level of accuracy : both should be penalised the first time this type of error occurs.
- 4.67 is incorrectly rounded – penalise on the first occurrence.

Note: All these "incorrect" answers may be assumed to come from 4.6789..., even if that value is not seen, but previous correct working is shown. However, 4.60 is wrong, as is 4.5, 4.8, and these should be penalised as being incorrect answers, not as examples of accuracy errors.

2. Alternative solutions

The points P, Q, R are three markers on level ground, joined by straight paths PQ, QR, PR as shown in the diagram. $\text{QR} = 9\text{ km}$, $\text{P}\hat{\text{Q}}\text{R} = 35^\circ$, $\text{P}\hat{\text{R}}\text{Q} = 25^\circ$.
(Note: in the original question, the first part was to find PR = 5.96)

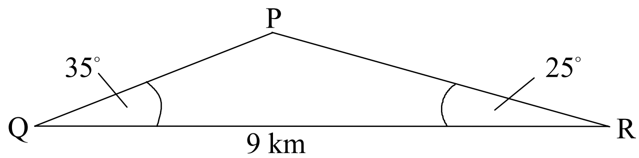
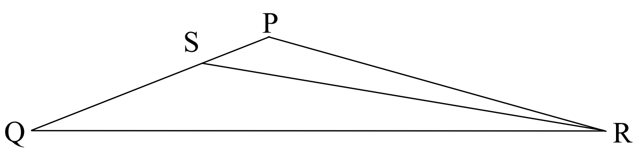


diagram not to scale

(a) Tom sets out to walk from Q to P at a steady speed of 8 km h^{-1}. At the same time, Alan sets out to jog from R to P at a steady speed of $a\text{ km h}^{-1}$. They reach P at the same time. Calculate the value of *a*. ***[7 marks]***

(b) The point S is on [PQ], such that $\text{RS} = 2\text{QS}$, as shown in the diagram.



Find the length QS. ***[6 marks]***

MARKSCHEME

(a) **EITHER**

Sine rule to find PQ

$PQ = \frac{9 \sin 25}{\sin 120}$ *(M1)(A1)*

$PQ = 4.39$ km *(A1)*

OR

Cosine rule: $PQ^2 = 5.96^2 + 9^2 - (2)(5.96)(9)\cos 25$ *(M1)(A1)*

$= 19.29$

$PQ = 4.39$ km *(A1)*

THEN

Time for Tom $= \frac{4.39}{8}$ *(A1)*

Time for Alan $= \frac{5.96}{a}$ *(A1)*

Then $\frac{4.39}{8} = \frac{5.96}{a}$ *(M1)*

$a = 10.9$ *(A1)* *(N5)*

[7 marks]

Note that the **THEN** part follows both **EITHER** and **OR** solutions, and this is shown by the alignment.

(b) **METHOD 1**

$RS^2 = 4QS^2$ *(A1)*

$4QS^2 = QS^2 + 81 - 18 \times QS \times \cos 35$ *(M1)(A1)*

$3QS^2 + 14.74QS - 81 = 0$ (or $3x^2 + 14.74x - 81 = 0$) *(A1)*

$\Rightarrow QS = -8.20$ or $QS = 3.29$ *(A1)*

therefore $QS = 3.29$ *(A1)*

METHOD 2

$\frac{QS}{\sin S\hat{R}Q} = \frac{2QS}{\sin 35}$ *(M1)*

$\Rightarrow \sin S\hat{R}Q = \frac{1}{2}\sin 35$ *(A1)*

$S\hat{R}Q = 16.7^\circ$ *(A1)*

Therefore, $Q\hat{S}R = 180 - (35 + 16.7) = 128.3^\circ$ *(A1)*

$\frac{9}{\sin 128.3} = \frac{QS}{\sin 16.7} \left(= \frac{SR}{\sin 35}\right)$ *(M1)*

$QS = \frac{9 \sin 16.7}{\sin 128.3} \left(= \frac{9 \sin 35}{2 \sin 128.3}\right) = 3.29$ *(A1)* *(N2)*

If candidates have shown no working, award ***(N5)*** for the correct answer 10.9 in part (a), and ***(N2)*** for the correct answer 3.29 in part (b). ***[6 marks]***

3. Follow through

Question

Calculate the acute angle between the lines with equations

$$\boldsymbol{r} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + s\begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{r} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Markscheme

Angle between lines = angle between direction vectors. (May be implied) ***(A1)***

Direction vectors are $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. (May be implied) ***(A1)***

$\begin{pmatrix} 4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left|\begin{pmatrix} 4 \\ 3 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right| \cos\theta$ ***(M1)***

$4 \times 1 + 3 \times (-1) = \sqrt{(4^2 + 3^2)}\sqrt{(1^2 + (-1)^2)} \cos\theta$ ***(A1)***

$\cos\theta = \frac{1}{5\sqrt{2}}$ (= 0.1414….) ***(A1)***

$\theta = 81.9^\circ$ (1.43 radians) ***(A1)*** ***(N3)***

Examples of solutions and marking

	Solutions	Marks allocated	
1.	$\begin{pmatrix} 4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left|\begin{pmatrix} 4 \\ 3 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right| \cos\theta$	***(A1)(A1)*** implied ***(M1)***	
	$\cos\theta = \frac{7}{5\sqrt{2}}$	***(A0)(A1)***	
	$\theta = 8.13^\circ$	(***A1***)**ft**	Total 5 marks
2.	$\cos\theta = \frac{\begin{pmatrix} 4 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \end{pmatrix}}{\sqrt{17}\sqrt{20}}$	***(A0)(A0)*** wrong vectors implied ***(M1)*** for correct method, ***(A1)*ft**	
	$= 0.2169$	***(A1)*ft**	
	$\theta = 77.5^\circ$	***(A1)*ft**	Total 4 marks
3.	$\theta = 81.9^\circ$	***(N3)***	Total 3 marks

Note that this candidate has obtained the correct answer, but not shown any working. The way the markscheme is written means that the first 2 marks may be implied by subsequent correct working, but the other marks are only awarded if the relevant working is seen. Thus award the first 2 implied marks, plus the final mark for the correct answer.

END OF EXAMPLES

1.

> **Note:** Some candidates may transpose the matrices and use row vectors. Do not penalize this method.

(a) $\begin{pmatrix} \cos k & -\sin k \\ \sin k & \cos k \end{pmatrix}\begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} -2 \\ 11 \end{pmatrix}$ ***(M1)(A1)***

$5\cos k - 10\sin k = -2$

$10\cos k + 5\sin k = 11$ ***(M1)***

$\cos k = 0.8$ and $\sin k = 0.6$ ***(A1)(A1)***

$\Rightarrow \boldsymbol{R} = \begin{pmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{pmatrix}$ (Accept answers given in trigonometric form) ***(A1)*** ***(N3)***

[6 marks]

(b) $\boldsymbol{T}^4 = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ***(M1)***

$\Rightarrow$ 4 successive applications of transformation $\boldsymbol{T}$ are equivalent to the identity transformation, which maps a figure onto itself. ***(R1)*** ***(N1)***

[2 marks]

(c) $\boldsymbol{Q} = \boldsymbol{TR} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{pmatrix}$ ***(M1)***

$= \begin{pmatrix} 2 & 1 \\ -1.4 & -0.2 \end{pmatrix}$ ***(AG)*** ***(N0)***

[1 mark]

(d) (i) $\begin{pmatrix} 2 & 1 \\ -1.4 & -0.2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ ***(M1)***

$2a + b = a$ and $-1.4a - 0.2b = b \Rightarrow a = b = 0$

The set of points which are mapped onto themselves is $\{(0, 0)\}$. ***(A1)*** ***(N1)***

(ii) $\begin{pmatrix} 2 & 1 \\ -1.4 & -0.2 \end{pmatrix}\begin{pmatrix} x \\ -x \end{pmatrix} = \begin{pmatrix} x \\ -1.2x \end{pmatrix}$ ***(M1)(A1)***

The image of the line $y = -x$ is the line $y = -\frac{6}{5}x$. ***(A1)*** ***(N1)***

[5 marks]

Total [14 marks]

2. (a) (i) $f'(x) = p\mathrm{e}^{px}(x+1) + \mathrm{e}^{px}$ ***(A1)***

$= \mathrm{e}^{px}\left(p(x+1)+1\right)$ ***(AG)*** ***(N0)***

(ii) The result is true for $n=1$ since

$\text{LHS} = \mathrm{e}^{px}\left(p(x+1)+1\right)$

and $\text{RHS} = p^{1-1}\mathrm{e}^{px}\left(p(x+1)+1\right) = \mathrm{e}^{px}\left(p(x+1)+1\right)$. ***(M1)***

Assume true for $n=k: f^{(k)}(x) = p^{k-1}\mathrm{e}^{px}\left(p(x+1)+k\right)$ ***(M1)***

$f^{(k+1)}(x) = \left(f^{(k)}(x)\right)' = p^{k-1}p\mathrm{e}^{px}\left(p(x+1)+k\right) + p^{k-1}\mathrm{e}^{px}p$ ***(M1)(A1)***

$= p^{k}\mathrm{e}^{px}\left(p(x+1)+k+1\right)$ ***(A1)***

Therefore, true for $n=k \Rightarrow$ true for $n=k+1$ and the proposition is proved by induction. ***(R1)***

[7 marks]

(b) (i) $f'(x) = \mathrm{e}^{\sqrt{3}x}\left(\sqrt{3}(x+1)+1\right) = 0$ ***(M1)***

$\Rightarrow x = -\dfrac{1+\sqrt{3}}{\sqrt{3}}\ \left(= -\dfrac{\sqrt{3}+3}{3}\right)$ ***(A1)*** ***(N1)***

(ii) $f''(x) = \sqrt{3}\mathrm{e}^{\sqrt{3}x}\left(\sqrt{3}(x+1)+2\right) = 0$ ***(M1)***

$\Rightarrow x = -\dfrac{2+\sqrt{3}}{\sqrt{3}}\ \left(= -\dfrac{2\sqrt{3}+3}{3}\right)$ ***(A1)*** ***(N1)***

[4 marks]

(c) $f(x) = \mathrm{e}^{0.5x}(x+1)$

EITHER

area $= -\int_{-2}^{-1} f(x)\,\mathrm{d}x + \int_{-1}^{2} f(x)\,\mathrm{d}x$ ***(M1)***

$= 8.08$ ***(A1)*** ***(N2)***

OR

area $= \int_{-2}^{2} \left| f(x) \right| \mathrm{d}x$ ***(M1)***

$= 8.08$ ***(A1)*** ***(N2)***

[2 marks]

Total [13 marks]

3. (a) (i) For points which lie in π_1 and π_2

(1) $2-2\lambda+\mu=2+s+t$
(2) $1+\lambda-3\mu=2s+t$
(3) $1+8\lambda-9\mu=1+s+t$ ***(M1)(A1)***

subtracting (3) from (1)

$1-10\lambda+10\mu=1$ ***(M1)***
$\Rightarrow \lambda=\mu$ ***(AG)*** ***(N0)***

(ii) On the line of intersection $\lambda=\mu$

$\Rightarrow$ an equation of the line is $\boldsymbol{r}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\8\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\-9\end{pmatrix}$ ***(M1)***

$=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\-2\\-1\end{pmatrix}$ ***(A1)*** ***(N1)***

[5 marks]

(b) The plane π_3 contains, e.g. the point (2, 0, –1). ***(A1)***

The equation of the plane is $\begin{pmatrix}x\\y\\z\end{pmatrix}\bullet\begin{pmatrix}3\\-2\\1\end{pmatrix}=\begin{pmatrix}2\\0\\-1\end{pmatrix}\bullet\begin{pmatrix}3\\-2\\1\end{pmatrix}=5.$ ***(M1)(A1)***

The cartesian equation of the plane is $3x-2y+z=5$. ***(A1)*** ***(N1)***

[4 marks]

(c) Intersection between line $\boldsymbol{r}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\-2\\-1\end{pmatrix}$ and π_3.

$3x-2y+z=5\Rightarrow 3(2-\lambda)-2(1-2\lambda)+1-\lambda=5$ ***(M1)(A1)***

This equation is satisfied by any real value of $\lambda\Rightarrow$ the 3 planes

intersect at the line $\boldsymbol{r}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\-2\\-1\end{pmatrix}$. ***(R1)*** ***(N1)***

[3 marks]

Total [12 marks]

4. (a) Let B be the random variable "diameter of the bolts produced by manufacturer B".

$\Rightarrow \mathrm{P}(B < 1.52) = 0.242$

$\Rightarrow \mathrm{P}\left(Z < \frac{1.52-\mu}{0.16}\right) = 0.242$ ***(M1)***

$\frac{1.52-\mu}{0.16} = -0.69988$ ***(A1)***

$\Rightarrow \mu = 1.63$ ***(A1)*** ***(N2)***

[3 marks]

(b) Let A be the random variable "diameter of the bolts produced by manufacturer A".

$\Rightarrow \mathrm{P}(A < 1.52) = \mathrm{P}\left(Z < \frac{1.52-1.56}{0.16}\right)$ ***(M1)***

$= \mathrm{P}(Z < -0.25) = 0.40129\ (0.4013)$ ***(A1)***

$\mathrm{P}(\text{diameter less than } 1.52\text{ mm}) = 0.44 \times 0.40129 + 0.56 \times 0.242$ ***(M1)(A1)***

$= 0.312\ \ (3\text{ s.f.})$ ***(AG)*** ***(N0)***

[4 marks]

(c) $\mathrm{P}\left(\text{bolt produced by B} \mid d < 1.52\right) = \frac{0.242 \times 0.56}{0.31209}$ ***(M1)(A1)***

$= 0.434$ ***(A1)*** ***(N2)***

[3 marks]

(d) $\mathrm{P}(B > 1.83) = \mathrm{P}\left(Z > \frac{1.83-1.63}{0.16}\right) = 0.10564$ ***(M1)(A1)***

$\mathrm{P}(1.52 < B < 1.83) = 1 - 0.242 - 0.10564$ ***(M1)***

$= 0.65236$ ***(A1)***

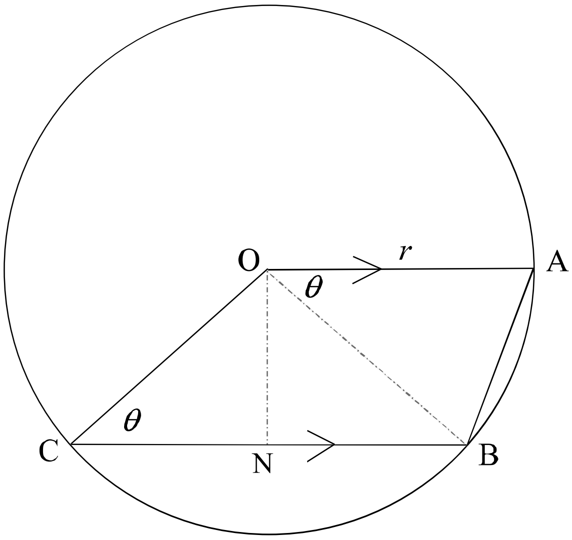
Expected gain $= \$\ 8000\left(0.242 \times (-0.85) + 0.65236 \times 1.50 + 0.10564 \times 0.50\right)$ ***(M1)***

$= \$\ 6605.28$ ***(A1)*** ***(N2)***

[6 marks]

Total [16 marks]

5.



(a) $h = r\sin\theta \quad \text{CB} = 2\text{CN} = 2r\cos\theta$ ***(A1)(A1)***

Using $T = (r + \text{CB})\dfrac{h}{2}$ ***(M1)***

$T = \dfrac{r^2}{2}(\sin\theta + 2\sin\theta\cos\theta)$ ***(A1)***

$= \dfrac{r^2}{2}(\sin\theta + \sin 2\theta)$ ***(AG)*** ***(N0)***

[4 marks]

(b) $\dfrac{dT}{d\theta} = \dfrac{r^2}{2}(\cos\theta + 2\cos 2\theta) = 0$ (for max) ***(M1)***

$\Rightarrow \cos\theta + 2(2\cos^2\theta - 1) = 4\cos^2\theta + \cos\theta - 2 = 0$ ***(M1)(AG)***

$\Rightarrow \cos\theta = 0.5931 \quad (\theta = 0.9359)$ ***(A1)***

$\dfrac{d^2T}{d\theta^2} = \dfrac{r^2}{2}(-\sin\theta - 4\sin 2\theta)$ ***(M1)***

$\theta = 0.9359 \quad \Rightarrow \quad \dfrac{d^2T}{d\theta^2} = -2.313r^2 < 0$

$\Rightarrow$ there is a **maximum** (when $\theta = 0.9359$) ***(R1)***

[5 marks]

(c) In triangle AOB: $\text{AB} = 2r\sin\dfrac{\theta}{2}$ ***(M1)(A1)***

Perimeter OABC $= 2r + 2r\cos\theta + 2r\sin\dfrac{\theta}{2} = 75$ ***(M1)***

When $\theta = 0.9359$, $r = 18.35$ cm ***(A1)***

Area OABC $= \dfrac{r^2}{2}(\sin\theta + \sin 2\theta) = \dfrac{18.35^2}{2}(\sin 0.9359 + \sin 1.872)$ ***(M1)***

$= 296 \text{ cm}^2$ ***(A1)*** ***(N3)***

[6 marks]

Total [15 marks]

6. (i) $X \sim \text{Po}(m)$

EITHER

$\text{P}(X \le 2) = \text{e}^{-m}\left(1 + m + \frac{m^2}{2}\right)$ ***(M1)***

$= 1 - 0.404 = 0.596$

$\Rightarrow m = 2.30$ ***(A1)***

$\text{P}(X < 2) = \text{e}^{-2.3}(1 + 2.3)$ ***(M1)***

$= 0.3309 = 0.331$ (3 s.f.) ***(A1)*** ***(N4)***

OR

Solving the equation $\text{cdf}\,(m,\,2) = 1 - 0.404 = 0.596$ ***(M1)***

$m = 2.30$ ***(A1)***

$\text{cdf}\,(2.3,\,1) = 0.3309 = 0.331$ (3 s.f.) ***(M1)(A1)*** ***(N4)***

[4 marks]

(ii) $\text{H}_0 : \frac{2}{3}$ of the balls are white. H_1 : proportion of white balls is not $\frac{2}{3}$. ***(A1)***

χ^2 is appropriate test distribution. ***(M1)***

Let X be the number of white balls, then $X \sim \text{B}\left(5, \frac{2}{3}\right)$ ***(M1)***

X	0	1	2	3	4	5
Observed values	8	9	52	78	70	26
Expected values	1	10	40	80	80	32

(A1)

Combining the first two columns.

X	0, 1	2	3	4	5
Observed values	17	52	78	70	26
Expected values	11	40	80	80	32

(A1)

Then $\chi^2_{\text{calc}} = \frac{36}{11} + \frac{144}{40} + \frac{4}{80} + \frac{100}{80} + \frac{36}{32} = 9.298$ ***(A1)***

The number of degrees of freedom is 4 so $\chi^2_{4,\,0.95} = 9.488$ ***(A1)***

Since $\chi^2_{\text{calc}} \le \chi^2_{4,\,0.95}$ we must accept hypothesis H_0. ***(R1)***

[8 marks]

continued…

Question 6 continued

(iii) (a) $H_0 : \mu_1 = \mu_2 \quad H_1 : \mu_1 \neq \mu_2$ ***(A1)***

[1 mark]

(b) **EITHER**

(i) $\sum_{i=1}^{12} x_i = 769.9; \sum_{i=1}^{12} y_i = 789.5 \qquad \bar{x} = 64.158; \bar{y} = 65.792$ ***(A1)***

The test statistic for the normal distribution is

$$z = \frac{\frac{789.5}{12} - \frac{769.9}{12}}{\sqrt{2^2\left(\frac{1}{12} + \frac{1}{12}\right)}}$$ ***(M1)(A1)***

$= 2.00$ (Accept ± 2.00) ***(A1)***

(ii) The critical value is 1.96. ***(A1)***
Hence we conclude that there is a difference in mean weight. ***(R1)***

OR

(i) $p = 0.0455$ (accept $p = 0.0454$) ***(A4)***

(ii) Since $p < 0.05$ (may be implied) ***(A1)***
we conclude that there is a difference in mean weight. ***(R1)***

[6 marks]

(c) **EITHER**

The level of significance is $2\left(1 - \Phi(2.00)\right) = 2(1 - 0.9773)$ ***(M1)***
$= 4.54\,\%$ ***(A1)***

OR

p-value $\Rightarrow 4.55\,\%$ level of significance (accept $4.54\,\%$) ***(A2)***

[2 marks]

(iv) (a) A t-distribution with 11 degrees of freedom must be used. ***(M1)***

$$s^2 = \frac{1}{n-1}\left(\sum (x_i - \bar{x})^2\right)$$

$$s^2 = \frac{1}{11}(99) = 9.00$$ ***(A1)***

$$t_{0.975,\,11} = 2.201$$ ***(A1)***

The confidence interval is $\bar{x} \pm \frac{2.201s}{\sqrt{12}}$ ***(M1)***

that is $[\bar{x} - 1.91, \bar{x} + 1.91]$ ***(A1)***

[5 marks]

(b) The length of the confidence interval is 5.38. ***(A1)***

Therefore $t_{\alpha,11} = \frac{5.38\sqrt{12}}{6} = 3.106$ ***(A1)***

$\alpha = 0.995$ ***(M1)***

Hence the level of confidence β is 99 % ***(A1)***

[4 marks]

Total [30 marks]

7. (i) (a) $A \# A = A' \cup A'$ ***(A1)***

$= A'$ ***(AG)***

[1 mark]

(b) $(A \# A) \# (B \# B) = A' \# B'$ ***(M1)***

$= (A')' \cup (B')'$ ***(A1)***

$= A \cup B$ ***(AG)***

[2 marks]

(c) $(A \# B) \# (A \# B) = (A' \cup B') \# (A' \cup B')$ ***(M1)(A1)***

$= (A' \cup B')'$ ***(A1)***

$= A \cap B$ (by de Morgan's law) ***(AG)***

[3 marks]

(ii) (a) $\gcd(a, a) = a > 1$, since $a \in S$. ***(A1)***

Hence R is reflexive. ***(AG)***

[1 mark]

(b) Since $\gcd(a, b) = \gcd(b, a)$, ***(M1)***

$\gcd(a, b) > 1 \Rightarrow \gcd(b, a) > 1$ ***(A1)***

Hence R is symmetric ***(AG)***

[2 marks]

(c) Any correct counter example e.g.

$\gcd(25, 15) = 5 \Rightarrow 25 R\ 15$ ***(A1)***

$\gcd(15, 21) = 3 \Rightarrow 15 R\ 21$ ***(A1)***

$\gcd(25, 21) = 1 \Rightarrow 25$ not R 21 ***(A1)***

Hence R is not transitive ***(AG)***

[3 marks]

(iii) (a) $a, b \in T \Rightarrow a * b \in \mathbb{R}$ ***(A1)***

if $a * b = 1$, $ab - a - b + 2 = 1$, $\Rightarrow ab - a - b + 1 = 0$ ***(M1)(A1)***

$\Rightarrow (a-1)(b-1) = 0 \Rightarrow a = 1$, or $b = 1$ contradiction ***(M1)(R1)***

so $a * b \in T$, i.e. closed ***(AG)***

[5 marks]

(b) $a * e = a \Rightarrow e(a-1) = 2(a-1) \Rightarrow e = 2$ (since $a \neq 1$) ***(M1)(A1)***

Hence 2 is the identity element for this operation. ***(A1)***

[3 marks]

continued ...

Question 7 (iii) continued

(c) (i) The formula is true for $n=1$ since $a=(a-1)^1+1$. ***(R1)***

Assume that it is true for $n=k$, i.e. $\overbrace{a*a*\cdots*a}^{k\ times}=(a-1)^k+1$ ***(M1)***

$\overbrace{a*a*\cdots*a}^{k+1\ times}=\left((a-1)^k+1\right)*a=\left((a-1)^k+1\right)a-\left((a-1)^k+1\right)-a+2$ ***(M1)***

$=(a-1)^k\times a+a-(a-1)^k-1-a+2$ ***(A1)***

$=(a-1)^k(a-1)+1$ ***(A1)***

$=(a-1)^{k+1}+1$

so the formula is proven by mathematical induction. ***(R1)***

[6 marks]

(ii) We require $a*a*...*a=2$ ***(M1)***

so that $(a-1)^n+1=2$ or $(a-1)^n=1$ ***(A1)***

Apart from $a=2$, the identity, the only solution is $a=0$. ***(A1)***

Since $0*0=2$, the element 0 has order 2. ***(A1)***

[4 marks]

Total [30 marks]

8. (i) (a) $656 = 2\times 272 + 112$ ***(M1)***

$272 = 2\times 112 + 48$

$112 = 2\times 48 + 16$ ***(A1)***

$48 = 3\times 16$

Therefore, $d = 16$. ***(A1)*** ***(N0)***

[3 marks]

(b) $112 = 656 - 2\times 272$ ***(M1)***

$48 = 272 - 2\times(656 - 2\times 272)$

$= 5\times 272 - 2\times 656$ ***(A1)***

$16 = 112 - 2\times 48$

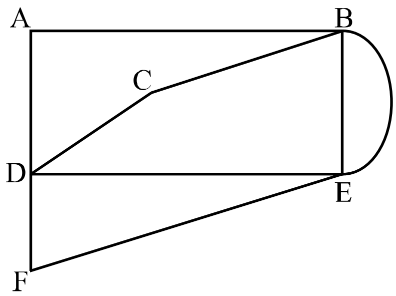
$= 656 - 2\times 272 - 2\times(5\times 272 - 2\times 656)$

$= -12\times 272 + 5\times 656 \quad (a = -12, b = 5)$. ***(A1)*** ***(N0)***

[3 marks]

(ii) (a) Any correct graph ***(A2)***

e.g.



[2 marks]

(b) All vertices are of even order (or equivalent). ***(R1)***

BEDABCDFEB (not unique). ***(A2)***

[3 marks]

(c) ABCDEF (not unique). ***(A1)***

[1 mark]

continued…

Question 8 continued

(iii) (a) Let $u_n = \lambda^n$.

$u_{n+2} - 2\lambda u_{n+1} + \lambda^2 u_n = \lambda^{n+2} - 2\lambda \times \lambda^{n+1} + \lambda^2 \times \lambda^n$ ***(M1)(A1)***

$= 0$ ***(AG)***

Let $u_n = n\lambda^n$.

$u_{n+2} - 2\lambda u_{n+1} + \lambda^2 u_n = (n+2)\lambda^{n+2} - 2\lambda(n+1)\lambda^{n+1} + \lambda^2 \times n\lambda^n$ ***(M1)***

$= n\lambda^{n+2}(1-2+1) + \lambda^{n+2}(2-2)$ ***(A1)***

$= 0$ ***(AG)***

Hence the general solution is $u_n = A\lambda^n + Bn\lambda^n$. ***(A1)***

[5 marks]

(b) Here, $\lambda = 3$ so the general solution is

$v_n = A3^n + Bn3^n$ ***(M1)***

$n = 1$ gives $3A + 3B = 2$ ***(A1)***

$n = 2$ gives $9A + 18B = 9$ ***(A1)***

The solution is $A = B = \frac{1}{3}$, so ***(A1)***

$v_n = \frac{1}{3}(n+1)3^n \left(\text{or } (n+1)3^{n-1}\right)$ ***(A1)***

[5 marks]

(iv)

Step	Vertices labelled	Working values	
1	P	P(0), Q–4, W–7	***(M1)***
2	P,Q	P(0), Q(4), W–6, R–16	***(A1)***
3	P,Q,W	P(0), Q(4), W(6), R–11, V–16	***(A1)***
4	P,Q,W,R	P(0), Q(4), W(6), R(11), V–15, U–14, S–21	***(A1)***
5	P,Q,W,R,U	P(0), Q(4), W(6), R(11), U(14), V–15, S–18, T–24	
6	P,Q,W,R,U,V	P(0), Q(4), W(6), R(11), U(14), V(15), S–18, T–24	***(A1)***
7	P,Q,W,R,U,V,S	P(0), Q(4), W(6), R(11), U(14), V(15), S(18), T–23	
8	P,Q,W,R,U,V,S,T	P(0), Q(4), W(6), R(11), U(14), V(15), S(18), T(23)	***(A1)***

Note: Accept any correct way of recording Dijkstra's algorithm.
Do **not** accept any other algorithm.

The length of the shortest path is 23. ***(A1)***
The shortest path is PQWRUST. ***(A1)***

[8 marks]

Total [30 marks]

9. (i) $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \ldots$ ***(A1)***

$e^{-x}\ln(1+x) = \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right)$ ***(M1)***

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - x^2 + \frac{x^3}{2} - \frac{x^4}{3} + \frac{x^3}{2} - \frac{x^4}{4} - \frac{x^4}{6}$

$= x - \frac{3x^2}{2} + \frac{4x^3}{3} - x^4 + \ldots$ ***(A1)(A1)(A1)(A1)***

[6 marks]

(ii) (a) $I \approx \frac{0.2}{2}\left(f(2) + 2f(2.2) + 2f(2.4) + 2f(2.6) + 2f(2.8) + 2f(3.0) + f(3.2)\right)$ ***(M1)(A1)***

$I \approx \frac{0.2}{2}\left(\frac{\ln 2}{2^2} + 2\left(\frac{\ln 2.2}{2.2^2} + \frac{\ln 2.4}{2.4^2} + \frac{\ln 2.6}{2.6^2} + \frac{\ln 2.8}{2.8^2} + \frac{\ln 3}{3^2}\right) + \frac{\ln 3.2}{3.2^2}\right)$ ***(A1)***

$= 0.170615654$

$= 0.171$ to 3 s.f. ***(A1)*** ***(N0)***

[4 marks]

(b) Let $f(x) = \frac{\ln x}{x^2}$

$f'(x) = x^{-3} - 2x^{-3}\ln x$ ***(A1)***

$f''(x) = -3x^{-4} - 2x^{-4} + 6x^{-4}\ln x \; (= -5x^{-4} + 6x^{-4}\ln x)$ ***(A1)***

Using the GDC, we see that $|f''(x)|$ is maximum when $x = 2$

and $|f''(2)| = 0.052569807$ (Accept bounds obtained by reasonable means) ***(A2)***

Dividing the interval into *n* sub-intervals, the error is less than

$\frac{1.2}{12} \times \left(\frac{1.2}{n}\right)^2 \times 0.0525\ldots$ ***(A1)***

We require $\frac{1.2}{12} \times \left(\frac{1.2}{n}\right)^2 \times 0.0525 \le 10^{-5}$ ***(M1)***

giving $n \ge 27.5$ ***(A1)***

Take $n = 28$ (Accept $n \ge 28$) ***(A1)***

[8 marks]

continued ...

Question 9 continued

(iii) $g(x)$ maximum when $g'(x) = 0$ ***(M1)***

$g'(x) = 3\sin^2 x\cos x - \sin x$ ***(A1)***

$g''(x) = -3\sin x + 9\cos^2 x\sin x - \cos x$ ***(A1)***

$x_1 = 1$, $x_{n+1} = x_n - \dfrac{g'(x_n)}{g''(x_n)}$ ***(M1)***

$x_2 = 1.35865477$

$x_3 = 1.222842884$

$x_4 = 1.206274735$

$x_5 = 1.205932648$

$x_6 = 1.205932499$

$x_7 = 1.205932499$ (at least 8 d.p. required) ***(A1)***

$g''(x)$ is negative in $[1, 1.5]$ ***(R1)***

Hence $g(x_6)$ is a maximum (and the only one in $[1, 1.5]$)

so $a = 1.205932499$ (accept 1.2059325) ***(A1)***

[7 marks]

(iv) $\dfrac{x^{k+1}\text{term}}{x^k\text{term}} = \left(\dfrac{(2k)!x^{k+1}}{k!(k+1)!}\right)\left(\dfrac{k!(k-1)!}{(2k-2)!x^k}\right)$ ***(M1)***

$= \left(\dfrac{2k(2k-1)}{k(k+1)}\right)x$ ***(A1)***

This $\to 4x$ (as $k \to \infty$) ***(A1)***

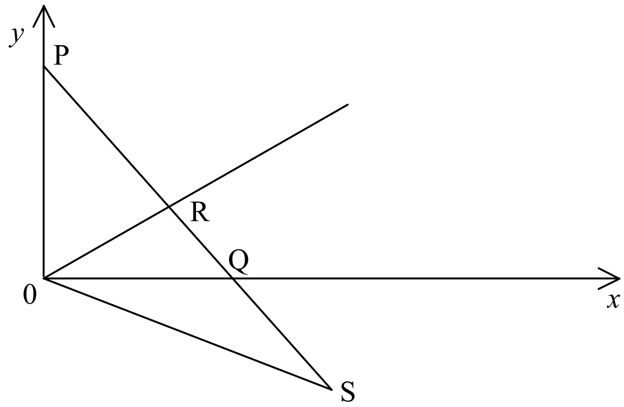
Using the ratio test, the series is convergent if $|4x| < 1$ ***(M1)***

So radius of convergence is $\dfrac{1}{4}$ (Accept $|x| < \dfrac{1}{4}$, but not $x < \dfrac{1}{4}$) ***(A1)***

[5 marks]

Total [30 marks]

10. (i)



Coordinates of P are (0, 12), Q is (4, 0) *(A1)*

Coordinates of R are found by solving $3x+y=12$ and $y-x=0$.

This gives (3, 3) *(A1)*

Coordinates of S are found by solving $3x+y=12$ and $y+x=0$.

This gives (6, –6) *(A1)*

$\text{PR}=\sqrt{90}$, $\text{RQ}=\sqrt{10}$ *(A1)*

$\text{PS}=\sqrt{360}$, $\text{QS}=\sqrt{40}$ *(A1)*

$\frac{\text{PR}}{\text{RQ}}=3$, $\frac{\text{PS}}{\text{SQ}}=-3$ *(A1)(A1)*

Note:	These ratios may be found by considering only the x or y coordinates of the points.

Harmonic division because ratios equal in magnitude, opposite in sign. *(R2)*

[9 marks]

(ii) (a) $y=m(x-1)$ *(A1)*

[1 mark]

(b) (i) Line meets parabola where $m^2(x-1)^2=2x$ *(M1)*

$m^2x^2-2(1+m^2)x+m^2=0$ *(A1)*

x-coordinates of U and V are $\frac{2(1+m^2)\pm\sqrt{4(1+m^2)^2-4m^4}}{2m^2}$ *(M1)(A1)*

x-coordinate of $\text{W}=\frac{1}{2}(\text{Sum of roots})=1+\frac{1}{m^2}$ *(A1)*

y-coordinate of $\text{W}=m\frac{1}{m^2}=\frac{1}{m}$ *(AG)*

(ii) Eliminating m, $x-1=y^2$ or $y^2=x-1$ *(M1)(A1)*

The focus is $\left(1+\frac{1}{4},0\right)$, i.e. $\left(\frac{5}{4},0\right)$ *(M1)(A1)*

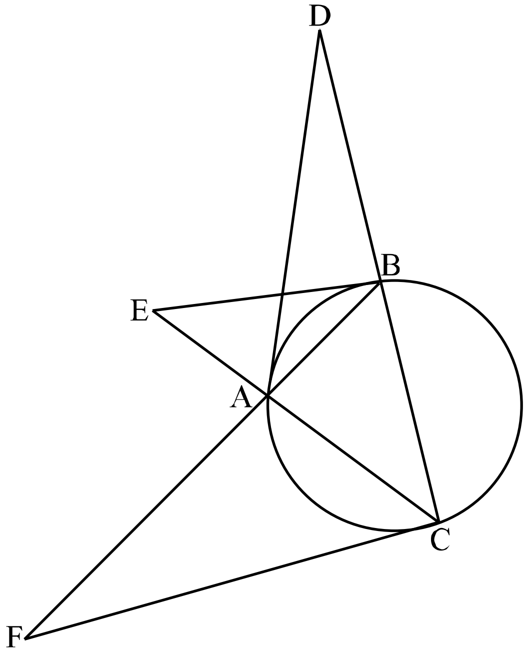
The directrix is $x=1-\frac{1}{4}$, i.e. $x=\frac{3}{4}$ *(A1)*

[10 marks]

continued…

Question 10 continued

(iii)



(a) Consider

"Ratio" $= \frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB}$

Using the tangent - secant theorem,

$DA^2 = DB \times DC$

$EB^2 = EC \times EA$ ***(M1)(A1)***

$FC^2 = FA \times FB$

so, ignoring signs, magnitude of "ratio" $= \frac{DA^2}{DC^2} \times \frac{EB^2}{EA^2} \times \frac{FC^2}{FB^2}$ ***(M1)(A1)***

Now ΔDAB is similar to ΔDCA, so ***(M1)***

$\frac{DA}{DC} = \frac{AB}{CA}$ ***(A1)***

Similarly, $\frac{EB}{EA} = \frac{BC}{AB}$ and $\frac{FC}{FB} = \frac{CA}{BC}$ ***(A1)***

Therefore, magnitude of "ratio" $= \frac{AB^2}{CA^2} \times \frac{BC^2}{AB^2} \times \frac{CA^2}{BC^2} = 1$ ***(M1)(A1)***

Since "Ratio" is clearly negative, it is equal to –1. ***(AG)***

[9 marks]

(b) It follows by the converse of Menelaus' Theorem that D, E and F are collinear. ***(R1)***

[1 mark]

Total [30 marks]