IB DIPLOMA PROGRAMME
PROGRAMME DU DIPLÔME DU BI
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M05/5/MATHL/HP2/ENG/TZ1/XX

22057205

MATHEMATICS
HIGHER LEVEL
PAPER 2

Wednesday 4 May 2005 (morning)

3 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.

Please start each question on a new page. You are advised to show all working, where possible. Where an answer is wrong, some marks may be given for correct method, provided this is shown by written working. Solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer.

SECTION A

*Answer all **five** questions from this section.*

1. *[Maximum mark: 13]*

A plane π_1 has equation $\boldsymbol{r} \bullet \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} = 15$.

(a) A point P $(p, -p, p)$ lies on plane π_1 and Q is the point where the plane π_1 meets the y-axis.

(i) Find the coordinates of P and of Q.

(ii) Show that $\overrightarrow{\mathrm{PQ}}$ is parallel to the vector $\boldsymbol{u}$, where $\boldsymbol{u} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$. *[5 marks]*

(b) Another plane π_2 intersects π_1 in the line (PQ). The point T $(1, 0, -1)$ lies on π_2.

(i) Find the equation of π_2, giving your answer in the form $Ax + By + Cz = D$.

(ii) Find the angle between π_1 and π_2. *[8 marks]*

2. *[Maximum mark: 14]*

(i) A transformation $\boldsymbol{A}$ maps $\mathrm{P}(2, -1)$ to $\mathrm{R}(1, 0)$ and $\mathrm{Q}(-4, -3)$ to $\mathrm{S}(0, 1)$.

(a) Find the matrix $\boldsymbol{A}$. *[3 marks]*

(b) (i) Find $\det \boldsymbol{A}$.

(ii) Hence find the area of Δ OPQ, where O is the origin. *[3 marks]*

(ii) Let $\boldsymbol{C} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and $\boldsymbol{D} = \frac{1}{13}\begin{pmatrix} -7 & 5 \\ 17 & 12 \end{pmatrix}$. Given that $\boldsymbol{D}$ is transformation $\boldsymbol{C}$ followed by transformation $\boldsymbol{M}$,

(a) find the matrix $\boldsymbol{M}$; *[3 marks]*

(b) describe fully the geometric transformation represented by $\boldsymbol{M}$. *[5 marks]*

3. *[Maximum mark: 13]*

(a) Use integration by parts to show that

$$\int 2x \arctan x \, \mathrm{d}x = (x^2+1)\arctan x - x + C, \text{ where } C \text{ is a constant.}$$ *[6 marks]*

(b) The probability density function of the random variable X is defined by

$$f(x) = \begin{cases} \dfrac{\pi}{2} - 2x\arctan x, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

The value of a is such that $\mathrm{P}(X < a) = \dfrac{3}{4}$.

(i) Show that a satisfies the equation $a(2\pi + 4) = 3 + 4(a^2+1)\arctan a$.

(ii) Find the value of a. *[7 marks]*

4. *[Maximum mark: 15]*

(i) Using mathematical induction, prove that

$$\sum_{r=1}^{n}(r+1)2^{r-1} = n(2^n) \text{ for all positive integers.}$$ *[7 marks]*

(ii) The first three terms of a geometric sequence are also the first, eleventh and sixteenth term of an arithmetic sequence.
The terms of the geometric sequence are all different.
The sum to infinity of the geometric sequence is 18.

(a) Find the common ratio of the geometric sequence, clearly showing all working. *[5 marks]*

(b) Find the common difference of the arithmetic sequence. *[3 marks]*

5. *[Maximum mark: 15]*

(a) The function g is defined by $g(x) = \dfrac{e^x}{\sqrt{x}}$, for $0 < x \le 3$.

(i) Sketch the graph of g.

(ii) Find $g'(x)$.

(iii) Write down an expression representing the gradient of the normal to the curve at any point. *[8 marks]*

(b) Let P be the point (x, y) on the graph of g, and Q the point $(1, 0)$.

(i) Find the gradient of (PQ) in terms of x.

(ii) Given that the line (PQ) is a normal to the graph of g at the point P, find the minimum distance from the point Q to the graph of g. *[7 marks]*

SECTION B

*Answer **one** question from this section.*

Statistics

6. *[Maximum mark: 30]*

(i) The random variable X has a standard deviation 3.5 and an unknown mean μ.

(a) A random sample of 100 observations was taken on X and it was found that $\sum x = 7204$. Determine a 95 % confidence interval for μ giving the limits correct to two decimal places. State the Central Limit Theorem and explain briefly why your solution requires the use of this theorem. *[5 marks]*

(b) Another random sample of 200 observations was taken on X and the following confidence interval was calculated.

$$[71.71, 72.75]$$

Find the confidence level of this interval. *[4 marks]*

(ii) John keeps two chickens, A and B, of different breeds. He wishes to determine whether or not the eggs laid by the two chickens have, on average, the same weight. He therefore weighs the next ten eggs laid by each chicken and he obtains the following results.

Chicken A eggs (x grams)	52.1	54.3	51.0	53.3	53.9	55.6	52.2	53.8	54.7	52.6
Chicken B eggs (y grams)	55.8	53.1	54.8	56.2	54.0	55.4	56.2	52.6	53.0	55.9

Assume that the weights of eggs laid by the two chickens are normally distributed with equal variances σ^2.

(a) Write down suitable hypotheses. *[1 mark]*

(b) Evaluate $\sum x, \sum x^2, \sum y$ and $\sum y^2$. Use your values to calculate the pooled unbiased estimate of σ^2. *[4 marks]*

(c) Determine an appropriate t-statistic to test your hypotheses. Using a significance level of 5 %, state whether or not the eggs laid by the two chickens have the same mean weight. *[4 marks]*

(This question continues on the following page)

(Question 6 continued)

(iii) A schoolgirl believes that the number of goals scored in soccer matches in a certain league can be modelled by a Poisson distribution. She consults the records for the past season and she obtains the following data from 132 matches.

Number of goals scored	0	1	2	3	4	5	6
Frequency	14	36	33	21	12	12	4

(a) Calculate the mean number of goals scored per match. *[2 marks]*

(b) Carry out a χ^2-test at the 10 % significance level to determine whether or not these data can be modelled by a Poisson distribution. *[10 marks]*

Sets, Relations and Groups

7. *[Maximum mark: 30]*

(i) The operation # defined on the set $\{a, b, c, d, e\}$ has the following operation table.

#	a	b	c	d	e
a	d	c	e	a	b
b	e	d	a	b	c
c	b	e	d	c	a
d	a	b	c	d	e
e	c	a	b	e	d

Show that only three of the four group axioms are satisfied. *[6 marks]*

(ii) Let R be a relation defined on 2×2 matrices such that, given the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{A}\ R\ \boldsymbol{B}$ if and only if there exists a matrix $\boldsymbol{H}$ such that $\boldsymbol{A} = \boldsymbol{HBH}^{-1}$.

(a) Show that R is an equivalence relation. *[5 marks]*

(b) Show that all matrices in the same equivalence class have equal determinants. *[2 marks]*

(c) Given the matrix $\boldsymbol{M} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, find a 2×2 matrix that is

(i) related to $\boldsymbol{M}$ (excluding $\boldsymbol{M}$ itself);

(ii) not related to $\boldsymbol{M}$. *[5 marks]*

(iii) Let $S = \{1, 2, 3, 4\}$ and f be a function, with domain and range S, defined by

$$f(x) = 2x(\text{modulo } 5).$$

(a) Prove that f is a bijection. *[3 marks]*

(b) Show that the composite function $f \circ f$ is its own inverse. *[3 marks]*

(iv) Let $(G, *)$ be a group, and H a subset of G. Given that for all $a, b \in H$, $a^{-1} * b \in H$, prove that $(H, *)$ is a subgroup of $(G, *)$. *[6 marks]*

Discrete Mathematics

8. *[Maximum mark: 30]*

(i) (a) Use the Euclidean algorithm to find the greatest common divisor, d, of 256 and 688. *[3 marks]*

(b) Hence express d in the form $256a + 688b$, where $a, b \in \mathbb{Z}$. *[3 marks]*

(ii) A graph G has adjacency matrix given by

$$\begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \\ \text{F} \end{array} \begin{array}{c} \begin{array}{cccccc} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} & \text{F} \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

(a) Draw the graph G. *[2 marks]*

(b) Explain why G has a Eulerian circuit. Find such a circuit. *[3 marks]*

(c) Find a Hamiltonian path. *[1 mark]*

(iii) The terms of the sequence $\{u_n\}$ satisfy the difference equation

$$u_{n+2} - 2\lambda u_{n+1} + \lambda^2 u_n = 0, \text{ where } n \in \mathbb{Z}^+ \text{ and } \lambda \in \mathbb{R}.$$

(a) Show that both $u_n = \lambda^n$ and $u_n = n\lambda^n$ satisfy this difference equation. Hence write down the general solution to this difference equation. *[5 marks]*

(b) The terms of the sequence $\{v_n\}$ satisfy the equation

$$v_{n+2} - 4v_{n+1} + 4v_n = 0 \text{ for } n \in \mathbb{Z}^+, \text{ and } v_1 = 1, v_2 = 3.$$

Find an expression for v_n in terms of n. *[5 marks]*

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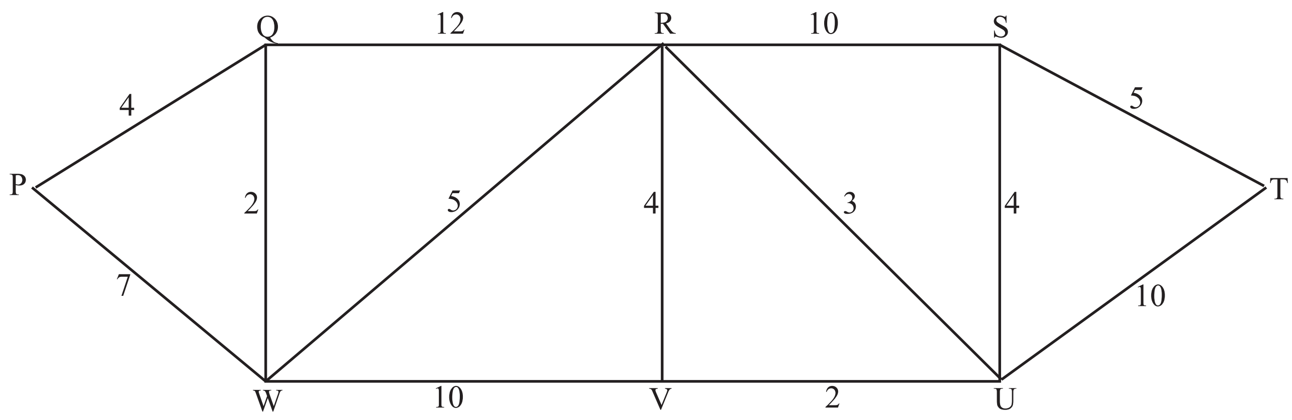
(Question 8 continued)

(iv) The following diagram shows a weighted graph.



Use Dijkstra's Algorithm to find the length of the shortest path between the vertices P and T. Show all the steps used by the algorithm and write down the shortest path. *[8 marks]*

Analysis and Approximation

9. *[Maximum mark: 30]*

(i) (a) Show that the series $\sum_{k=1}^{\infty} \frac{(k-1)}{k!}$ is convergent. *[3 marks]*

(b) Find the sum of the series. *[3 marks]*

(ii) The function f is defined by $f(x) = e^{-x} \cos 2x$.

(a) (i) Show that $f''(x) + 2f'(x) + 5f(x) = 0$.

(ii) Show that $f^{(n+2)}(0) + 2f^{(n+1)}(0) + 5f^{(n)}(0) = 0$, for $n \in \mathbb{Z}^+$. *[6 marks]*

(b) Find the Maclaurin series for $f(x)$ up to and including the term in x^4. *[4 marks]*

(iii) Consider the equation $\ln(2x) = \sin\left(\frac{x}{2}\right)$ where $0.5 < x < 1$.

(a) Sketch a graph to show that this equation has exactly one root. *[2 marks]*

The equation can be rearranged in two ways as follows.

$$x = h(x) \text{ where } h(x) = \frac{1}{2} e^{\sin\left(\frac{x}{2}\right)}$$

$$x = g(x) \text{ where } g(x) = 2\arcsin\left(\ln(2x)\right)$$

(b) Obtain expressions for $h'(x)$ and $g'(x)$. *[4 marks]*

(c) (i) Show that, for $0.5 < x < 1$, $|h'(x)| < 1$ and $|g'(x)| > 1$.

(ii) Hence state, with a reason, which of these rearrangements leads to a convergent fixed-point iteration. *[5 marks]*

(d) Use this convergent fixed-point iteration with $x_0 = 0.7$ to solve the equation $\ln(2x) = \sin\left(\frac{x}{2}\right)$. Give your answer correct to **six** decimal places. *[3 marks]*

Euclidean Geometry and Conic Sections

10. *[Maximum mark: 30]*

(i) The line $2x+y=6$ meets the y-axis at P, the x-axis at Q, the line $y-x=0$ at R and the line $y+x=0$ at S. Show that P, Q, R, S divide the line segment [PS] in a harmonic ratio. *[9 marks]*

(ii) The straight line l of gradient m passes through the point (2, 0).

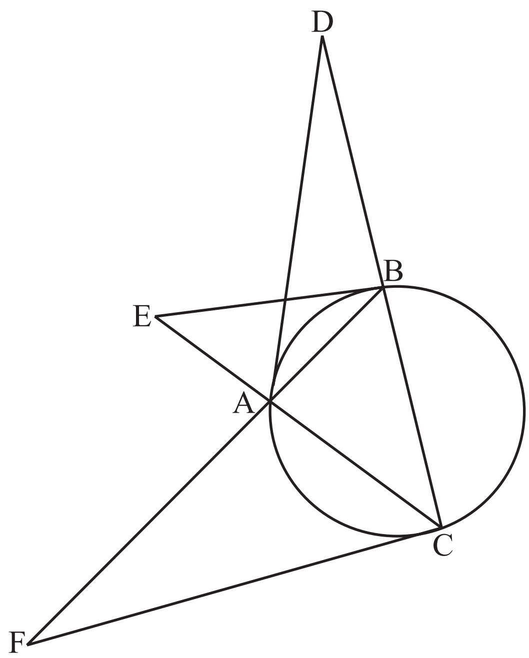
(a) Write down the equation of l. *[1 mark]*

(b) A parabola has equation $y^2 = 4x$. The line l meets the parabola at U and V. The midpoint of [UV] is denoted by W.

(i) Show that the coordinates of W are $\left(2+\frac{2}{m^2}, \frac{2}{m}\right)$.

(ii) Show that the locus of W as m varies is a parabola. Find the coordinates of its focus and the equation of its directix. *[10 marks]*

(iii)



The figure shows a triangle ABC inscribed in a circle. The tangent to the circle at A meets (BC) at D, the tangent at B meets (CA) at E and the tangent at C meets (AB) at F.

(a) Show that $\frac{\text{BD}}{\text{DC}} \times \frac{\text{CE}}{\text{EA}} \times \frac{\text{AF}}{\text{FB}} = -1.$ *[9 marks]*

(b) State briefly, with a reason, what conclusion can be drawn from this result. *[1 mark]*