MARKSCHEME

May 2004

MATHEMATICS

Higher Level

Paper 2

29 pages

Paper 2 Markscheme

Instructions to Examiners

Note: Where there are 2 marks (*e.g.* M2, A2) for an answer do NOT split the marks unless otherwise instructed.

1 Method of marking

(a) All marking must be done using a **red** pen.

(b) Marks should be noted on candidates' scripts as in the markscheme:

- show the breakdown of individual marks using the abbreviations ***(M1)***, ***(A2)*** *etc*., unless a part is completely correct;
- write down each part mark total, indicated on the markscheme (for example, ***[3 marks]***) – it is suggested that this be written at the end of each part, and underlined;
- write down and circle the total for each question at the end of the question.

2 Abbreviations

The markscheme may make use of the following abbreviations:

(M) Marks awarded for **Method**

(A) Marks awarded for an **Answer** or for **Accuracy**

(N) Marks awarded for correct answers, if **no** working shown: they may not be all the marks for the question. Examiners should only award these marks for correct answers where there is no working.

(R) Marks awarded for clear **Reasoning**

(AG) **Answer Given** in the question and consequently marks are **not** awarded

Note: In general, it is not possible to award ***(M0)(A1)***.

Examiners should use **(d)** to indicate where discretion has been used. It should only be used for decisions on follow through and alternative methods. It must be accompanied by a brief note to explain the decision made

Follow through (ft) marks should be awarded where a correct method has been attempted but error(s) are made in subsequent working which is essentially correct.

- Penalize the error when it first occurs
- Accept the incorrect result as the appropriate quantity in all subsequent working
- If the question becomes much simpler then use discretion to award fewer marks
- Use **(d)** to indicate where discretion has been used. It should only be used for decisions on follow through and alternative methods. It must be accompanied by a brief note to explain the decision made.

3 Using the Markscheme

(a) This markscheme presents a particular way in which each question may be worked and how it should be marked. **Alternative methods** have not always been included. Thus, if an answer is wrong then the working must be carefully analysed in order that marks are awarded for a different method in a manner which is consistent with the markscheme. Indicate the awarding of these marks by **(d)**.

Where alternative methods for complete questions or parts of questions are included, they are indicated by **METHOD 1**, **METHOD 2**, *etc*. Other alternative part solutions are indicated by **EITHER….OR**. It should be noted that ***G*** marks have been removed, and GDC solutions will not be indicated using the **OR** notation as on previous markschemes.

Candidates are expected to show working on this paper, and examiners should not award full marks for just the correct answer. Where it is appropriate to award marks for correct answers with no working, it will be shown on the markscheme using the ***N*** notation. All examiners will be expected to award marks accordingly in these situations.

(b) Unless the question specifies otherwise, accept **equivalent forms**. For example: $\frac{\sin\theta}{\cos\theta}$ for $\tan\theta$. On the markscheme, these equivalent numerical or algebraic forms will generally be written in brackets after the required answer. Paper setters will indicate the required answer, by allocating full marks at that point. Further working should be ignored, even if it is incorrect. For example: if candidates are asked to factorize a quadratic expression, and they do so correctly, they are awarded full marks. If they then continue and find the roots of the corresponding equation, do not penalize, even if those roots are incorrect, *i.e.* once the correct answer is seen, ignore further working.

(c) As this is an international examination, all **alternative forms of notation** should be accepted. For example: 1.7, $1\cdot 7$, 1,7; different forms of vector notation such as $\vec{u}$, $\bar{u}$, $\underline{u}$; $\tan^{-1} x$ for arctan *x*.

4 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

There are two types of accuracy error. Candidates should be penalized **once only IN THE PAPER** for an accuracy error **(AP)**.

Award the marks as usual then write –1**(AP)** against the answer and also on the **front** cover

Rounding errors: only applies to final answers not to intermediate steps.

Level of accuracy: when this is not specified in the question the general rule *unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures* applies.

- If a final correct answer is incorrectly rounded, apply the **AP**
  OR
- If the level of accuracy is not specified in the question, apply the **AP** for answers not given to 3 significant figures. (Please note that this has changed from May 2003).

Note: If there is no working shown, and answers are given to the correct two significant figures, apply the **AP**. However, do **not** accept answers to one significant figure without working.

5 Graphic Display Calculators

Many candidates will be obtaining solutions directly from their calculators, often without showing any working. They have been advised that they must use mathematical notation, not calculator commands when explaining what they are doing. Incorrect answers without working will receive no marks. However, if there is written evidence of using a graphic display calculator correctly, method marks may be awarded. Where possible, examples will be provided to guide examiners in awarding these method marks.

Calculator penalties

Candidates are instructed to write the make and model of their calculator on the front cover. Please apply the following penalties where appropriate.

(a) Illegal calculators

If candidates note that they are using an illegal calculator, please report this on a PRF, and deduct 10% of their overall mark. Note this on the front cover. The most common examples are:

Texas Instruments: TI-89 (plus); TI-92 (plus); TI-Voyage 200
Casio: *fx9970*; *fx2.0* algebra; classpad
HP: 38–95 series

(b) Calculator box not filled in.

Please apply a calculator penalty ***(CP)*** of 1 mark if this information is not provided. Note this on the front cover.

Examples

1 Accuracy

A question leads to the answer 4.6789….

- 4.68 is the correct 3 s.f. answer.
- 4.7, 4.679 are to the wrong level of accuracy: both should be penalized the first time this type of error occurs.
- 4.67 is incorrectly rounded – penalize on the first occurrence.

Note: All these “incorrect” answers may be assumed to come from 4.6789…, even if that value is not seen, but previous correct working is shown. However, 4.60 is wrong, as is 4.5, 4.8, and these should be penalized as being incorrect answers, not as examples of accuracy errors.

**2	Alternative solutions**

Question

The points P, Q, R are three markers on level ground, joined by straight paths PQ, QR, PR as shown in the diagram. $\text{QR} = 9 \text{ km}$, $\text{P}\hat{\text{Q}}\text{R} = 35^\circ$, $\text{P}\hat{\text{R}}\text{Q} = 25^\circ$.
(Note: in the original question, the first part was to find PR = 5.96)

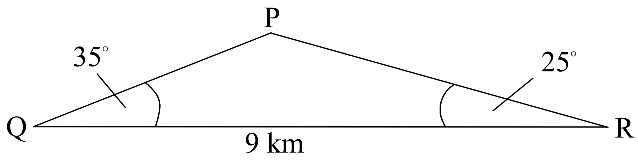
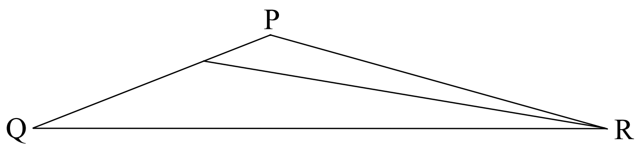


diagram not to scale

(a)	Tom sets out to walk from Q to P at a steady speed of 8 km h^{-1}. At the same time, Alan sets out to jog from R to P at a steady speed of $a \text{ km h}^{-1}$. They reach P at the same time. Calculate the value of *a*. ***[7 marks]***

(b)	The point S is on [PQ], such that $\text{RS} = 2\text{QS}$, as shown in the diagram.



Find the length QS. ***[6 marks]***

Markscheme

(a)	**EITHER**

Sine rule to find PQ

$\text{PQ} = \dfrac{9\sin 25}{\sin 120}$	***(M1)(A1)***

$\text{PQ} = 4.39 \text{ km}$	***(A1)***

OR

Cosine rule: $\text{PQ}^2 = 5.96^2 + 9^2 - (2)(5.96)(9)\cos 25$	***(M1)(A1)***

$= 19.29$

$\text{PQ} = 4.39 \text{ km}$	***(A1)***

THEN

Time for Tom $= \dfrac{4.39}{8}$	***(A1)***

Time for Alan $= \dfrac{5.96}{a}$	***(A1)***

Then $\dfrac{4.39}{8} = \dfrac{5.96}{a}$	***(M1)***

$a = 10.9$	***(A1)***	***(N5)***

[7 marks]

Note that the **THEN** part follows both **EITHER** and **OR** solutions, and this is shown by the alignment.

(b) **METHOD 1**

$\text{RS}^2 = 4\text{QS}^2$ ***(A1)***

$4\text{QS}^2 = \text{QS}^2 + 81 - 18 \times \text{QS} \times \cos 35$ ***(M1)(A1)***

$3\text{QS}^2 + 14.74\text{QS} - 81 = 0$ (or $3x^2 + 14.74x - 81 = 0$) ***(A1)***

$\Rightarrow \text{QS} = -8.20$ or $\text{QS} = 3.29$ ***(A1)***

therefore QS = 3.29 ***(A1)*** ***(N2)***

METHOD 2

$$\frac{\text{QS}}{\sin \text{S}\hat{\text{R}}\text{Q}} = \frac{2\text{QS}}{\sin 35}$$ ***(M1)***

$$\Rightarrow \sin \text{S}\hat{\text{R}}\text{Q} = \frac{1}{2}\sin 35$$ ***(A1)***

$\text{S}\hat{\text{R}}\text{Q} = 16.7^\circ$ ***(A1)***

Therefore, $\text{Q}\hat{\text{S}}\text{R} = 180 - (35 + 16.7) = 128.3^\circ$ ***(A1)***

$$\frac{9}{\sin 128.3} = \frac{\text{QS}}{\sin 16.7} \left(= \frac{\text{SR}}{\sin 35} \right)$$ ***(M1)***

$$\text{QS} = \frac{9 \sin 16.7}{\sin 128.3} \left(= \frac{9 \sin 35}{2 \sin 128.3} \right) = 3.29$$ ***(A1)*** ***(N2)***

If candidates have shown no working award ***(N5)*** for the correct answer 10.9 in part (a) and ***(N2)*** for the correct answer 3.29 in part (b).

[6 marks]

3 Follow through

Question

Calculate the acute angle between the lines with equations

$$\boldsymbol{r} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + s\begin{pmatrix} 4 \\ 3 \end{pmatrix} \text{ and } \boldsymbol{r} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Markscheme

Angle between lines = angle between direction vectors. (May be implied.)	***(A1)***	
Direction vectors are $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. (May be implied.)	***(A1)***	
$\begin{pmatrix} 4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left\|\begin{pmatrix} 4 \\ 3 \end{pmatrix}\right\| \left\|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\| \cos\theta$	***(M1)***	
$4 \times 1 + 3 \times (-1) = \sqrt{(4^2 + 3^2)}\sqrt{(1^2 + (-1)^2)}\cos\theta$	***(A1)***	
$\cos\theta = \dfrac{1}{5\sqrt{2}}$ $(= 0.1414\ldots)$	***(A1)***	
$\theta = 81.9^\circ$ (1.43 radians)	***(A1)***	***(N3)***

Examples of solutions and marking

	Solutions	**Marks allocated**
1.	$\begin{pmatrix} 4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left\|\begin{pmatrix} 4 \\ 3 \end{pmatrix}\right\| \left\|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\| \cos\theta$	***(A1)(A1)*** implied ***(M1)***
	$\cos\theta = \dfrac{7}{5\sqrt{2}}$	***(A0)(A1)***
	$\theta = 8.13^\circ$	***(A1)*ft**
		Total 5 marks
2.	$\cos\theta = \dfrac{\begin{pmatrix} 4 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \end{pmatrix}}{\sqrt{17}\sqrt{20}}$	***(A0)(A0)*** wrong vectors implied ***(M1)*** for correct method, ***(A1)*ft**
	$= 0.2169$	***(A1)*ft**
	$\theta = 77.5^\circ$	***(A1)*ft**
		Total 4 marks
3.	$\theta = 81.9^\circ$	***(N3)***
		Total 3 marks

Note that this candidate has obtained the correct answer, but not shown any working. The way the markscheme is written means that the first 2 marks may be implied by subsequent correct working, but the other marks are only awarded if the relevant working is seen. Thus award the first 2 implied marks, plus the final mark for the correct answer.

END OF EXAMPLES

1.

Note:	If candidates consistently pre-multiply by a row vector, and obtain their answers as row vectors, award marks as shown in the markscheme. In part (a)(i), this gives $\boldsymbol{M} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

(a) (i) Let $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ***(M1)***

Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}; \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$,

$\Rightarrow a + 2b = 2; c + 2d = 3$;

$4a + 5b = 5; 4c + 5d = 6$

Solving these equations gives

$\boldsymbol{M} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$ (Accept $a = 0, b = 1, c = -1, d = 2$) ***(A1)(A1)(A1)(A1)*** ***(N4)***

(ii) $\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ ***(A1)***

Image of A' is $(3, 4)$. ***(A1)*** ***(N2)***

Note:	Award ***(A0)(A0)*** for using the translation $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to give (3, 4).

[7 marks]

(b) Let $\boldsymbol{T} = \begin{pmatrix} e \\ f \end{pmatrix}$. Then $\begin{pmatrix} e \\ f \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \Rightarrow e + 1 = 2$ and $f + 3 = 2$ ***(M1)***

$\Rightarrow e = 1$ and $f = -1$. $\boldsymbol{T} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. ***(A1)*** ***(N2)***

[2 marks]

(c)

Notes:	In both parts (i) and (ii), award no marks if ***T*** in part (b) is not a vector, or if they multiply by ***T***.

(i) $\boldsymbol{T} : \begin{pmatrix} 5 \\ 7 \end{pmatrix} \to \begin{pmatrix} 6 \\ 6 \end{pmatrix}$ ***(A1)***

$\boldsymbol{M} : \begin{pmatrix} 6 \\ 6 \end{pmatrix} \to \begin{pmatrix} 6 \\ 6 \end{pmatrix}$, image of D is (6, 6) ***(A1)*** ***(N2)***

(ii) $\boldsymbol{M} : \begin{pmatrix} 5 \\ 7 \end{pmatrix} \to \begin{pmatrix} 7 \\ 9 \end{pmatrix}$ ***(A1)***

$\boldsymbol{T} : \begin{pmatrix} 7 \\ 9 \end{pmatrix} \to \begin{pmatrix} 8 \\ 8 \end{pmatrix}$, image of D is (8, 8) ***(A1)*** ***(N2)***

Note:	Award ***(A0) (A1)*ft, *(A0) (A1)*ft** if candidates reverse the order in **both** parts (i) and (ii).

[4 marks]

Total [13 marks]

2. (i) (a) Probability that Jack wins on his first throw $= \frac{2}{3}$ (or 0.667). *(A1)*

[1 mark]

(b) Probability that Jill wins on her first throw: $\frac{1}{3} \times \frac{2}{3}$ *(M1)*

$= \frac{2}{9}$ (or 0.222). *(A1)* *(N2)*

[2 marks]

(c) **EITHER**

Probability that Jack wins the game:

$\left(\frac{2}{3}\right) + \left(\frac{1}{3} \times \frac{1}{3} \times \frac{2}{3}\right) + \ldots$ *(M1)*

$= \frac{2}{3} \times \frac{1}{1 - \frac{1}{9}}$ *(A1)*

$= \frac{3}{4}$ *(A1)* *(N2)*

OR

If p is the probability that Jack wins the game then

$p = \frac{2}{3} + \frac{1}{3} \times \frac{1}{3} p$, *(M1)*

so that $p = \frac{\frac{2}{3}}{1 - \frac{1}{9}}$ *(A1)*

$= \frac{3}{4}$ *(A1)* *(N2)*

[3 marks]

continued…

Question 2 continued

(ii) (a) $k\int_0^2 x^2 \mathrm{d}x = 1$ ***(M1)***

$\Rightarrow k\left[\frac{x^3}{3}\right]_0^2 = \frac{k8}{3} = 1$ ***(A1)***

$\Rightarrow k = \frac{3}{8}$ ***(AG)***

[2 marks]

(b) (i) $\mathrm{E}(X) = \frac{3}{8}\int_0^2 x.x^2 \mathrm{d}x \quad \left(= \frac{3}{8}\left[\frac{x^4}{4}\right]_0^2 \right)$ ***(M1)***

$= \frac{3}{2}$ ***(A1)*** ***(N2)***

(ii) The median m must be a number such that

$\frac{3}{8}\int_0^m x^2 \mathrm{d}x = \frac{1}{2} \quad \left(\text{or} \frac{3}{8}\int_m^2 x^2 \mathrm{d}x\right)$ ***(M1)(A1)***

$\frac{3}{8}\left[\frac{x^3}{3}\right]_0^m = \frac{3}{8}\left(\frac{m^3}{3} - 0\right) = \frac{1}{2}$ ***(A1)***

$\frac{m^3}{8} = \frac{1}{2} \Rightarrow m^3 = 4$.

$\Rightarrow m = \sqrt[3]{4}$ $(= 1.59 \text{ to } 3 \text{ s.f.})$. ***(A1)*** ***(N4)***

[6 marks]

Total [14 marks]

3. (a) **METHOD 1**

$|\boldsymbol{u}| = |\boldsymbol{v}| = 1$. Let ϕ be the acute angle betweeen $\boldsymbol{u}$ and $\boldsymbol{u}+\boldsymbol{v}$, and ψ the acute angle between $\boldsymbol{v}$ and $\boldsymbol{u}+\boldsymbol{v}$.

Then $\cos\phi = \dfrac{\boldsymbol{u}\cdot(\boldsymbol{u}+\boldsymbol{v})}{|\boldsymbol{u}||\boldsymbol{u}+\boldsymbol{v}|} = \dfrac{1+\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}+\boldsymbol{v}|}$ ***(M1)(A1)***

$\cos\psi = \dfrac{\boldsymbol{v}\cdot(\boldsymbol{u}+\boldsymbol{v})}{|\boldsymbol{v}||\boldsymbol{u}+\boldsymbol{v}|} = \dfrac{\boldsymbol{v}\cdot\boldsymbol{u}+1}{|\boldsymbol{u}+\boldsymbol{v}|} = \dfrac{1+\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}+\boldsymbol{v}|} = \cos\phi$ ***(A1)***

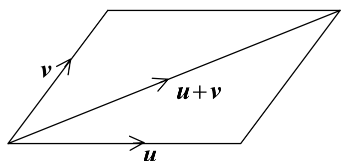
$\cos\phi = \cos\psi$

$\Rightarrow \theta = \psi$ ***(A1)***

$\Rightarrow \boldsymbol{u}+\boldsymbol{v}$ bisects the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ ***(AG)***

METHOD 2

Geometric approach



(M1)

$\boldsymbol{u}+\boldsymbol{v}$ is the diagonal of the parallelogram ***(A1)***

$|\boldsymbol{u}| = |\boldsymbol{v}| \Rightarrow$ rhombus ***(A1)***

$\Rightarrow$ Diagonal bisects the angle between the sides ***(A1)***

$\Rightarrow \boldsymbol{u}+\boldsymbol{v}$ bisects the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ ***(AG)***

[4 marks]

(b) $\vec{\text{BA}} = \boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}$, $\vec{\text{BC}} = 4\boldsymbol{i}+2\boldsymbol{j}+4\boldsymbol{k}$ ***(A1)***

The corresponding unit vectors are $\boldsymbol{u} = \frac{1}{3}(\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k})$, $\boldsymbol{v} = \frac{1}{3}(2\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k})$ ***(M1)***

The line l bisects $\boldsymbol{u}$ and $\boldsymbol{v}$, so its direction is parallel to $\boldsymbol{u}+\boldsymbol{v}$, *i.e.* $\boldsymbol{i}+\boldsymbol{j}+\frac{4}{3}\boldsymbol{k}$. ***(A1)***

An equation for l is

$\boldsymbol{r} = (\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k}) + \lambda\left(\boldsymbol{i}+\boldsymbol{j}+\frac{4}{3}\boldsymbol{k}\right)$. $\left(\text{Accept } x-1 = y-3 = \frac{3(z-2)}{4}.\right)$ ***(A1)***

Note: Award ***(A1)*****ft** for finding an equation without using unit vectors.

[4 marks]

(c) An equation of (AC) is $\boldsymbol{r} = (5\boldsymbol{i}+5\boldsymbol{j}+6\boldsymbol{k}) + \mu(3\boldsymbol{i}+2\boldsymbol{k})$. ***(M1)(A1)***

The coordinates of D must satisfy both equations. Thus:

$1+\lambda = 5+3\mu;\ 3+\lambda = 5;\ 2+\frac{4}{3}\lambda = 6+2\mu$. ***(M1)(A1)***

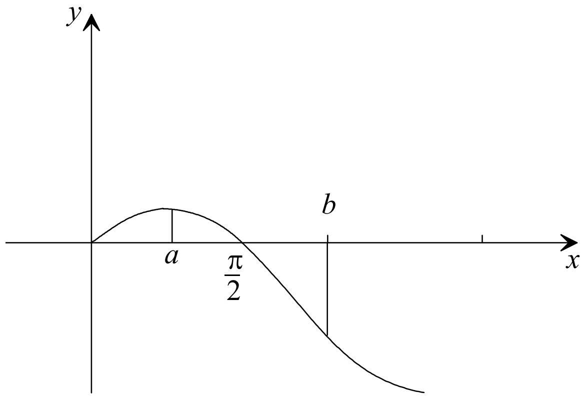
$\mu = -\frac{2}{3}$ and/or $\lambda = 2$. ***(A1)***

The coordinates of D are $\left(3, 5, \frac{14}{3}\right)$ ***(A1)*** ***(N3)***

[6 marks]

Total [14 marks]

4. (a)



(A1)(A1)

Note:	Award ***(A1)*** for the general shape and ***(A1)*** for a and b in approximately correct position with relation to $\frac{\pi}{2}$. (Accept a and b marked on the graph. Also accept b at the origin.)

[2 marks]

(b) (i) $a = 0.860$ ***(A1)***

(ii) The derivative of the function is $f'(x) = \cos x - x\sin x$. ***(M1)***

$b = 2.29$ (Also accept $b = 0$.) ***(A1)*** ***(N2)***

[3 marks]

(c) $\int x\cos x\,dx = x\sin x - \int \sin x\,dx$ ***(M1)(A1)***

$= x\sin x + \cos x + C$ ***(A1)***

[3 marks]

(d) $\int_0^{\frac{\pi}{2}} x\cos x\,dx = [x\sin x + \cos x]_0^{\frac{\pi}{2}}$ ***(M1)***

$= \frac{\pi}{2} - 1$ ***(A1)***

Note:	Award ***(A0)*** for a GDC answer of 0.57079…

[2 marks]

Total [10 marks]

5. (a) $\cos(A+B)=\cos A\cos B-\sin A\sin B$, $\cos(A-B)=\cos A\cos B+\sin A\sin B$ ***(M1)(A1)***

Hence $\cos(A+B)+\cos(A-B)=2\cos A\cos B$ ***(AG)***

[2 marks]

(b) (i) $T_1(x)=\cos(\arccos x)$ ***(M1)***

$=x$ ***(A1)***

(ii) $T_2(x)=\cos(2\arccos x)$ ***(A1)***

$=2\cos^2(\arccos x)-1$ ***(A2)***

$=2x^2-1$ ***(AG)***

[5 marks]

(c) (i) $T_{n+1}(x)+T_{n-1}(x)=\cos\left[(n+1)\arccos x\right]+\cos\left[(n-1)\arccos x\right]$ ***(A1)***

Using part (a) with $A=n\arccos x,\ B=\arccos x$ ***(M1)***

$T_{n+1}(x)+T_{n-1}(x)=2\cos(n\arccos x)\cos(\arccos x)$ ***(A1)***

$=2x\cos(n\arccos x)$ ***(A1)***

$=2xT_n(x)$ ***(AG)***

(ii) Let P_n be the statement: $T_n(x)$ is a polynomial of degree, $n\in\mathbb{Z}^+$

$T_1(x)=x$, a polynomial of degree one. ***(A1)***

So P_1 is true.

$T_2(x)=2x^2-1,$ is a polynomial of degree two. ***(A1)***

So P_2 is true.

Assume that P_k is true. ***(M1)***

From part (c)(i), $T_{k+1}(x)=2xT_k(x)-T_{k-1}(x)$ ***(M1)***

Assume P_{k-1} is true as well. ***(M1)***

$T_k(x)$ has degree k

$\Rightarrow 2xT_k(x)$ has degree $(k+1)$ ***(A1)***

and as $T_{k-1}(x)$ has degree $(k-1)$

$\Rightarrow T_{k+1}(x)$ has degree $(k+1)$ ***(A1)***

By the principle of mathematical induction, P_n is true for all positive integers *n*. ***(R1)***

Notes:	These arguments may be in a different order. There is a maximum of 6 marks in part (ii) for candidates who do not consider a two stage process.

[12 marks]

Total [19 marks]

6.

> **Note:** In this question do not penalize accuracy for more than 3 s.f.

(i) (a) $\mathrm{P}(X=2)=\mathrm{e}^{-3}\times\dfrac{3^2}{2!}=0.224$ ***(M1)(A1)*** ***(N2)***

[2 marks]

(b) $\mathrm{P}(X=2\mid X>0)=\dfrac{\mathrm{P}(X=2)}{\mathrm{P}(X>0)}$ ***(M1)***

$=\dfrac{0.224041\ldots}{1-\mathrm{e}^{-3}}$ ***(A1)***

$=0.236$ ***(A1)*** ***(N3)***

[3 marks]

(ii) $\bar{x}=18.1875;\ {s_{15}}^2=4.2638\ldots$ ***(A1)(A1)***

Variance is unknown so t-distribution must be used with DF 15. ***(M1)***

Confidence limits are

$$18.1875\pm\frac{2.131\times\sqrt{4.2638}}{4}$$ ***(M1)(A1)***

giving $[17.1, 19.3]$ ***(A1)*** ***(N6)***

> **Note:** Award a maximum of 3 marks for use of normal distribution, award ***(A1)(A1)(M0)(M0)(A0)(A1)*****ft** for $[17.2, 19.2]$.

[6 marks]

continued...

Question 6 continued

(iii) (a) The alternative hypothesis is that "it is not the case that six of the coins are fair and the other one has two heads". ***(A2)***

Note: Award ***(A2)*** if H_0 and H_1 are both complementary and exclusive. Award ***(A1)*** if if H_0 and H_1 are mutually exclusive but not complementary.

[2 marks]

(b) Use of χ^2-test. ***(M1)***
Under H_0, the number of heads is $X+1$, where $X \sim B(6, 0.5)$. ***(M1)***
We obtain the following table:

Number of heads	Observed frequency	Expected frequency
1	12	**5**
2	43	**30**
3	79	**75**
4	86	**100**
5	65	**75**
6	29	**30**
7	6	**5**

(A2)

Note: Award ***(A1)*** if one error in expected frequency, ***(A0)*** if more than one error. This table may be inferred from answers below.

$$\chi^2_{\text{calc}} = \frac{(12-5)^2}{5} + \ldots + \frac{(6-5)^2}{5} = 19.2$$ ***(A1)***

EITHER

DF = 6 : Critical value = 12.592 ***(A1)***
As $12.596 < 19.2$ reject H_0. ***(R1)*** ***(N0)***

OR

p-value = 0.00384 (accept answers in [0.00384 , 0.00389]) ***(A1)***

As 0.00384 < 0.05 reject H_0. ***(R1)*** ***(N0)***

[7 marks]

continued...

Question 6 continued

(iv) **METHOD 1**

This is a two-sample t-test *(M1)*
with DF 15. *(A1)*

$\bar{x}_m = 173.44, \bar{x}_w = 160.125$ *(A1)(A1)*

$s^2_{\text{pooled}} = 33.7\ (33.673)$ *(A2)*

$$t = \frac{173.444 - 160.125 - 10}{\sqrt{33.673}\sqrt{\frac{1}{9}+\frac{1}{8}}} = 1.18$$ *(M1)(A1)*

Note: Award ***(A8)*** for a correct t-value.

Critical value = 1.341 *(A1)*

As 1.341 > 1.18, we conclude that the mean height of men does not exceed the mean height of women by more than 10 cm. *(R1)* *(N0)*

METHOD 2

p-value = 0.129 (accept 0.125 to 0.129) *(A9)*
As 0.129 > 0.10, we conclude that the mean height of men does not exceed the mean height of women by more than 10 cm. *(R1)* *(N0)*

[10 marks]

Total [30 marks]

7. (i) (a) Reflexivity: $(x_1, y_1)R(x_1, y_1)$ since $x_1 + y_1 = x_1 + y_1$ ***(A1)***

Symmetry: $(x_1, y_1)R(x_2, y_2) \Rightarrow (x_2, y_2)R(x_1, y_1)$ since $x_1 + y_2 = x_2 + y_1$

$\Rightarrow x_2 + y_1 = x_1 + y_2$ ***(A1)***

Transitivity: Suppose that $(x_1, y_1)R(x_2, y_2)$ and $(x_2, y_2)R(x_3, y_3)$. Then,

$x_1 + y_2 = x_2 + y_1$ and $x_3 + y_2 = x_2 + y_3$ ***(M1)***

Subtracting, $x_1 - x_3 = y_1 - y_3$ or $x_1 + y_3 = x_3 + y_1$ ***(A1)***

It follows that $(x_1, y_1)R(x_3, y_3)$.

[4 marks]

(b) $x_1 + y_2 = x_2 + y_1 \Rightarrow y_1 - x_1 = y_2 - x_2$ ***(M1)***

The equivalence classes are lines with equations $y = x + \text{Constant}$. ***(A1)***

[2 marks]

(ii)

Note:	As * is clearly commutative, there is no need to check **both** left **and** right identity in (a), or **both** left **and** right inverse in (b).

(a) $x * e = x \Rightarrow xe - x - e + 2 = x$ ***(M1)***

$e(x-1) = 2x - 2 \Rightarrow e = 2$ ***(A1)***

[2 marks]

(b) Assume inverse element of 3 is I

$3 * I = 3I - 3 - I + 2 = 2$ ***(M1)***

$\Rightarrow I = \dfrac{3}{2}$ ***(A1)***

[2 marks]

(c) (i) $(x * y) * z = (xy - x - y + 2) * z$. ***(A1)***

$= xyz - xz - yz + 2z - xy + x + y - 2 - z + 2$ ***(A1)***

$xyz - yz - zx - xy + x + y + z$ ***(AG)***

(ii) $x * (y * z) = x * (yz - y - z + 2)$ ***(A1)***

$= xyz - xy - xz + 2x - x - yz + y + z - 2 + 2$ ***(A1)***

$= (x * y) * z$ ***(A1)***

It follows that $*$ is associative. ***(R1)***

[6 marks]

continued…

Question 7 continued

(iii) (a) (i) $3 \otimes 5 = 15$ ***(A1)***

(ii) $3 \otimes 7 = 5$ ***(A1)***

(iii) $9 \otimes 11 = 3$ ***(A1)***

[3 marks]

(b) (i) The operation table is

$\otimes$	**1**	**3**	**5**	**7**	**9**	**11**	**13**	**15**
1	1	3	5	7	9	11	13	15
3	3	9	**15**	**5**	11	1	7	13
5	5	**15**	9	3	13	7	1	11
7	7	**5**	3	1	15	13	11	9
9	9	11	13	15	1	**3**	5	7
11	11	1	7	13	**3**	9	15	5
13	13	7	1	11	5	15	9	3
15	15	13	11	9	7	5	3	1

(A2)

> **Note:** Award ***(A2)*** if the circled numbers are correct, ***(A1)*** if 3 or 4 are correct, ***(A0)*** otherwise. The bold numbers were found in part (a)

(ii) Closure: The table shows that no new elements are generated. ***(R1)***
Identity: 1 is the identity. ***(R1)***
Inverse: Every row and column has a "1". ***(R1)***
(Associative given).
So (S, $\otimes$) is a group. ***(AG)***

[5 marks]

(c) (i) Elements of order 2 are 7, 9, 15. ***(A2)***

> **Note:** Award ***(A1)*** if one correct element is given.

(ii) Elements of order 4 are 3, 5, 11, 13. ***(M1)(A1)***

> **Note:** If no working shown, award ***(M1)(A0)*** if one correct element is given.

[4 marks]

(d) Using 3 as the generator, a sub-group of order 4 is $\{1, 3, 9, 11\}$. ***(M1)(A1)***

> **Note:** Another possibility is $\{1, 5, 9, 13\}$.

[2 marks]

Total [30 marks]

8. (i) The characteristic equation is $r^2 - 3r - 28 = 0$ *(M1)*

so that $r = -4$, or $r = 7$ *(A1)*

The general solution is $x_n = A(-4)^n + B(7)^n$ *(A1)*

Setting $n = 0, 1$ we obtain

$$A + B = 7\ ,\ -4A + 7B = -6$$ *(M1)*

$A = 5, B = 2$

So $x_n = 5(-4)^n + 2(7)^n$ (Accept $A = 5$, $B = 2$.) *(A1)*

[5 marks]

(ii) (a) (i) A bipartite graph is a graph whose vertices can be divided into two sets and in which edges always connect a vertex from one set to a vertex from the other set. *(A2)*

Note: Accept equivalent definitions, *e.g.* chromatic number = 2.

(ii) An isomorphism between two graphs M and N is a one-to-one correspondence between vertices which maps the adjacency matrix of M onto the adjacency matrix of N. *(A1)* *(A1)*

Note: This definition can be simplified when M and N are simple graphs.

[4 marks]

(b) Let M contain a cycle $(x_1, x_2); (x_2, x_3); \ldots (x_n, x_1)$ *(M1)*

Let Φ denote an isomorphism between M and N.

Consider $\Phi(x_1, x_2); \Phi(x_2, x_3); \ldots \Phi(x_n, x_1)$ *(M1)*

The adjacency property shows that this is a cycle, proving the result. *(R1)*

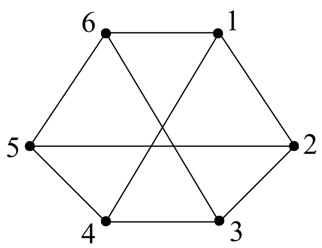
Note: Accept less formal proofs in which the above statements are made verbally rather than mathematically.

[3 marks]

continued...

Question 8 continued

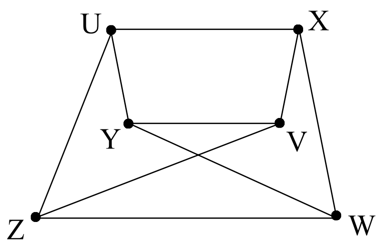
(c) (i)



The graph is bipartite **(A1)**

because by putting $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$, the conditions in (ii) (a) (i) are satisfied. **(R1)**

(ii)



The graphs are isomorphic **(A1)**

Consider the mapping

$1 \to \text{U},\ 2 \to \text{X},\ 3 \to \text{V},\ 4 \to \text{Y},\ 5 \to \text{W},\ 6 \to \text{Z}$ **(M1)**

Since adjacency is preserved, the graphs are isomorphic. **(R1)**

Note: Other isomorphisms exist – please check any answer carefully.

(iii) The graphs are not isomorphic **(A1)**

because

EITHER

J contains a cycle or order 3 but *H* does not. **(R1)**

OR

H is bipartite but *J* is not **(R1)**

[7 marks]

continued...

(Question 8 continued)

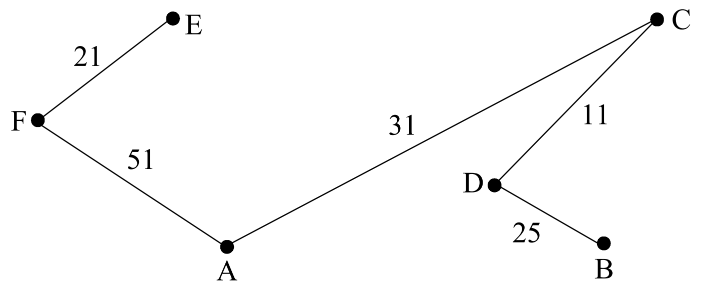
(iii) (a) Let S be a set of vertices and T be a set of edges. The algorithm is organised as follows:

Choice	Edge	Weight	T	S	
1	CD	11	{CD}	{C,D}	***(M1)(A1)***
2	EF	21	{CD, EF}	{C,D,E,F}	
3	BD	25	{CD, EF, BD}	{C,D,E,F,B}	
4	AC	31	{CD, EF, BD, AC}	{A,B,C,D,E,F}	***(A1)***
5	AF	51	{CD, EF, BD, AC, AF,}	{A,B,C,D,E,F}	***(A1)***

Note: Accept any solution which indicates the order in which the edges are selected.

[4 marks]

(b) The minimal spanning tree is (weights not needed)



(A1)

Total weight = 139 ***(A1)***

[2 marks]

(iv) (a) Every non-empty set of positive integers (or subset of $\mathbb{Z}^+$) contains a least element. ***(A1)(A1)***

Note: Award ***(A1)*** for "non-empty" set and ***(A1)*** for "positive integers or ($\mathbb{Z}^+$)".

[2 marks]

(b) Assume that the statement is not true, *i.e.* for some integers $a, b, na < b$ for every positive integer n. Let

$$S = \{b - na \mid n \text{ is a positive integer}\}$$ ***(M1)***

Then S consists entirely of positive integers. By the well-ordering principle, S has a least element, say $b - ma$. ***(M1)***
But $b-(m+1)a < b - ma$ showing that $b - ma$ is not the smallest element in S, resulting in a contradiction. Hence the given statement is true. ***(R1)***

[3 marks]

Total [30 marks]

9. **Note:** In this question do not penalize accuracy for more than 3 s.f.

(i) Use of comparison test *(M1)*

$\cos\left(\frac{(n-1)\pi}{2n}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{2n}\right) = \sin\left(\frac{\pi}{2n}\right)$ *(A1)*

$\lim_{n\to\infty}\left(\frac{\sin\left(\frac{\pi}{2n}\right)}{\frac{\pi}{2n}}\right) = 1$ *(M1)(A1)*

As $\sum_{n=1}^{\infty}\frac{\pi}{2n}$ is divergent, the given series is also divergent. *(R1)*

[5 marks]

(ii) (a) $A = \int_0^{\frac{\pi}{2}} \sin x\,\mathrm{d}x = -[\cos x]_0^{\frac{\pi}{2}}$ *(M1)*

$= 1$ *(A1)*

[2 marks]

(b) The 4th derivative of $f(x)$ is $\sin x$ with maximum value 1 in the given interval. *(A1)*

$\left|\text{Simpson error term}\right| \le \frac{\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2n}\right)^4}{180} \times 1 = \frac{\left(\frac{\pi}{2}\right)^5}{180n^4}$ *(M1)*

We require $\frac{\left(\frac{\pi}{2}\right)^5}{180n^4} \le 10^{-4}$ *(A1)*

implying that $n \ge 4.8$ *(A1)*

so take $n = 6$ (must be even) *(M1)*

The values are

$y_0 = 0$
$y_1 = 0.2588190451$
$y_2 = 0.5$
$y_3 = 0.7071067812$
$y_4 = 0.8660254038$
$y_5 = 0.9659258263$
$y_6 = 1$

Using Simpson's rule,

$A = \frac{\pi}{36}(4\times 0.2588\ldots + 2\times 0.5 + 4\times 0.7071\ldots + 2\times 0.8660\ldots + 4\times 0.9659\ldots + 1)$

$= 1.000026312$ *(A1)* *(N2)*

Note: If candidates take an odd value of n, do not award the final 2 marks in part (b) or the ***(A1)*** in part (c).

[6 marks]

(c) Error $= 0.000026312 < 10^{-4}$ *(A1)(AG)*

[1 mark]

continued...

Question 9 continued

(iii) (a) **METHOD 1**

In the interval $\left[0, \frac{\pi}{2}\right]$, $\sin x$ and its derivative are continuous. By the mean value theorem, for $x > 0$, there exists c in $[0, x]$ such that

$$\frac{\sin x}{x} = \cos c\,.$$ ***(M1)***

Since $\sin x$ and $\cos c$ are positive in the interval and $\cos c \le 1$, it follows that $\sin x \le x$. ***(A1)***

METHOD 2

In the interval $\left[0, \frac{\pi}{2}\right]$, $\cos t$ and its derivative are continuous. By the mean value theorem for integrals, there exists c in $[0, x]$ such that

$$\cos c = \frac{1}{x}\int_0^x \cos t \, \mathrm{d}t = \frac{\sin x}{x}$$ ***(M1)***

Since $\sin x$ and $\cos c$ are positive in the interval and $\cos c \le 1$, it follows that $\sin x \le x$. ***(A1)***

Note: Award ***(M1)(A0)*** if the mean value theorem is applied to $\left[0, \frac{\pi}{2}\right]$

[2 marks]

continued...

Question 9 (iii) continued

(b) **METHOD 1**

Consider, for x in $\left[0, \frac{\pi}{2}\right]$,

$$x - \sin x = \int_0^x (1 - \cos t)\,\mathrm{d}t = \int_0^x 2\sin^2\left(\frac{t}{2}\right)\mathrm{d}t \qquad \textit{(M1)}$$

$$\leq \int_0^x 2\left(\frac{t}{2}\right)^2 \mathrm{d}t \text{ (using the result in part (a))} \qquad \textit{(A1)}$$

$$= \frac{x^3}{6} \qquad \textit{(A1)}$$

It follows that

$$\sin x \geq x - \frac{x^3}{6} \qquad \textit{(A1)}$$

METHOD 2

The McLaurin series for $\sin x$ is $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$ ***(A1)***

The series is alternating**.** ***(A1)***

The general term is decreasing in absolute value

since $\dfrac{\frac{x^{2n+1}}{(2n+1)!}}{\frac{x^{2n-1}}{(2n-1)!}} = \dfrac{x^2}{4n^2 + 2n} \leq \dfrac{\pi^2}{4} \cdot \dfrac{1}{6} < 1$ ***(A1)***

Hence $x - \sin x \leq \dfrac{x^3}{3!}\left(= \dfrac{x^3}{6}\right)$ ***(A1)***

[4 marks]

continued…

Question 9 continued

(iv) (a) **METHOD 1**

We replace a sum over all integers $\le 4m$ by a sum over all odd numbers $\le 4m$ plus a sum over all even numbers $\le 4m$: ***(M1)(A1)***

$$S_{4m} = \sum_{k=1}^{2m} \frac{\sin\left[(2k-1)\frac{\pi}{2}\right]}{2k-1+\sin\left[(2k-1)\frac{\pi}{2}\right]} + \sum_{k=1}^{2m} \frac{\sin k\pi}{2k+\sin k\pi}$$

$$= \sum_{k=1}^{2m} \frac{\sin\left(k\pi - \frac{\pi}{2}\right)}{2k-1+\sin\left(k\pi - \frac{\pi}{2}\right)} \qquad \textit{(M1)(A1)}$$

since all the terms in the second sum are zero.
Now replace a sum over all integers $\le 2m$ by a sum over all odd numbers $\le 2m$ plus a sum over all even numbers $\le 2m$: ***(A1)***

$$S_{4m} = \sum_{k=1}^{m} \frac{(-1)^{2k-2}}{4k-3+(-1)^{2k-2}} + \sum_{k=1}^{m} \frac{(-1)^{2k-1}}{4k-1+(-1)^{2k-1}} \qquad \textit{(M1)}$$

$$= \sum_{k=1}^{m} \frac{1}{4k-2} - \sum_{k=1}^{m} \frac{1}{4k-2} = 0 \qquad \textit{(A1)}$$

METHOD 2

Writing $S_n = \sum_{k=1}^{n} a_k$

Terms $a_1 = \frac{1}{2}$, $a_2 = 0$, $a_3 = -\frac{1}{2}$, $a_4 = 0$ ***(M1)***

$S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2}$, $S_3 = 0$, $S_4 = 0$ ***(M1)***

All the even terms $(a_{2m}, m > 0)$ are zero ***(M1)***

(For any $m \ge 0$) $a_{4m+1} = \frac{1}{4m+2}$ ***(A1)***

$a_{4m+3} = -\frac{1}{4m+2}$ ***(A1)***

$a_{4m+1} + a_{4m+2} + a_{4m+3} + a_{4m+4} = 0$ ***(A1)***

$S_4 = 0$ and $S_{4m} = 0$ ***(R1)***

[7 marks]

continued...

Question 9 (iv) continued

(b) Let $n \geq 1$.

If $n = 4m$, then $S_n = 0$ so $S_n \leq \dfrac{1}{n}$

If $n = 4m+1$, then $S_n = \dfrac{1}{(4m+2)} \leq \dfrac{1}{n}$

If $n = 4m+2$, then $S_n = \dfrac{1}{(4m+2)} \leq \dfrac{1}{n}$

If $n = 4m+3$, then $S_n = \dfrac{1}{(4m+2)} - \dfrac{1}{(4m+2)} = 0 \leq \dfrac{1}{n}$

In all cases, $0 \leq S_n \leq \dfrac{1}{n}$ so $S_n \to 0$ as $n \to \infty$. ***(M1)(A1)***

[2 marks]

(c) Since the limit of the partial sums of the given series is zero, the series is convergent with limit zero. ***(R1)***

[1 mark]

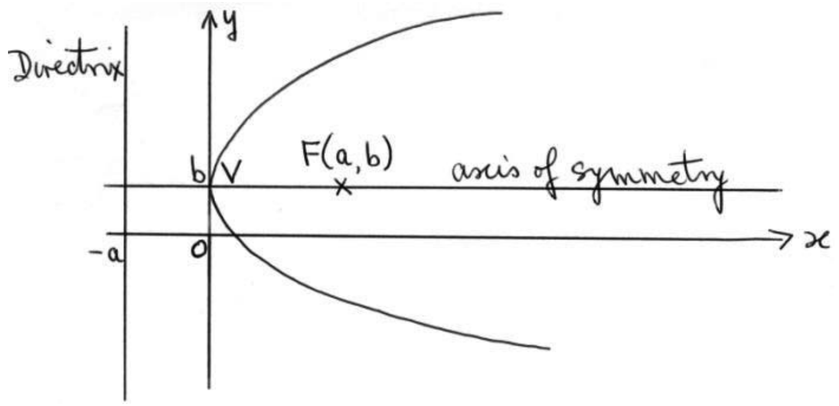
Total [30 marks]

10. (i) (a) (i) Let (x, y) be on C. Then, using the focus-directrix definition,

$$(x-a)^2 + (y-b)^2 = (x+a)^2 \qquad \textit{(M1)}$$

$$(y-b)^2 = (x+a)^2 - (x-a)^2 = 4ax \qquad \textit{(A1)}$$

(ii)

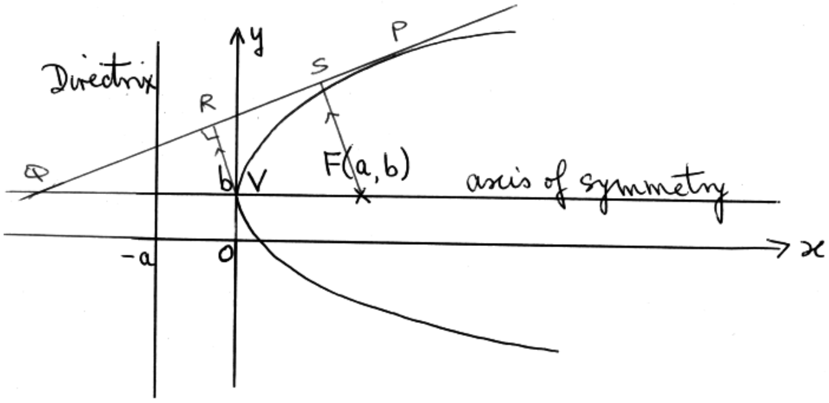


(A1)(A1)(A1)(A1)

Note:	Award ***(A1)*** for graph, ***(A1)*** for vertex, ***(A1)*** for axis of symmetry and ***(A1)*** for focus and directrix.

[6 marks]

(b)



$$x = \frac{3a}{2} \Rightarrow (y-b)^2 = 6a^2 \Rightarrow y = a\sqrt{6} + b$$

So P is the point $\left(\frac{3a}{2}, a\sqrt{6} + b\right)$. ***(A1)***

We now need the gradient of the tangent:

$$2(y-b)\frac{dy}{dx} = 4a$$

At P, $\frac{dy}{dx} = \frac{2}{\sqrt{6}}$ ***(M1)(A1)***

Equation of PQ is

$$y - a\sqrt{6} - b = \frac{2}{\sqrt{6}}\left(x - \frac{3a}{2}\right) \qquad \text{(I)} \qquad \textit{(M1)(A1)}$$

To find the coordinates of Q, put $y = b$.

continued…

Question 10 (i) (b) continued

This gives $x = -\frac{3a}{2}$ so Q is $\left(-\frac{3a}{2}, b\right)$. ***(A1)***

The gradient of RV is $-\frac{\sqrt{6}}{2}$. ***(A1)***

The equation of RV is

$$y - b = -\frac{\sqrt{6}}{2}x \qquad \text{(II)}$$ ***(A1)***

The coordinates of R are found by solving (I) and (II) simultaneously.

It follows that the x-coordinate of R is $-\frac{3a}{5}$. ***(M1)(A1)***

$$\frac{\text{PR}}{\text{RQ}} = \frac{\left(\frac{3a}{2} + \frac{3a}{5}\right)}{\left(-\frac{3a}{5} + \frac{3a}{2}\right)} = \frac{7}{3}$$ ***(M1)(A1)***

[12 marks]

(c) The equation of FS is

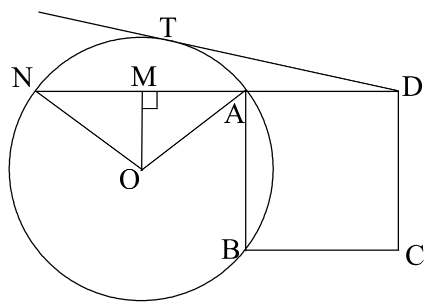
$$y - b = -\frac{\sqrt{6}}{2}(x - a) \qquad \text{(III)}$$ ***(M1)(A1)***

The coordinates of S are found by solving (I) and (III) simultaneously.

This gives $\text{S}\left(0, \frac{a\sqrt{6}}{2} + b\right)$. ***(A1)(A1)***

[4 marks]

(ii)



(A2)

$\text{AM} = \sqrt{r^2 - \frac{a^2}{4}}$ so $\text{AN} = 2\sqrt{r^2 - \frac{a^2}{4}}$ ***(M1)(A1)***

Use the result $\text{DT}^2 = \text{DA} \times \text{DN}$. ***(M1)***

$$(2a)^2 = a \times \left(a + 2\sqrt{r^2 - \frac{a^2}{4}}\right)$$ ***(A1)***

leading to $r = \frac{\sqrt{10}a}{2}$. ***(M1)(A1)***

[8 marks]

Total [30 marks]