MARKSCHEME

May 2004

MATHEMATICS

Higher Level

Paper 2

Paper 2 Markscheme

Instructions to Examiners

Note: **Where there are 2 marks (*e.g.* M2, A2) for an answer do NOT split the marks unless otherwise instructed.**

1   Method of marking

(a)   All marking must be done using a **red** pen.

(b)   Marks should be noted on candidates' scripts as in the markscheme:

 • show the breakdown of individual marks using the abbreviations *(M1)*, *(A2) etc*., unless a part is completely correct;
 • write down each part mark total, indicated on the markscheme (for example, *[3 marks]*) – it is suggested that this be written at the end of each part, and underlined;
 • write down and circle the total for each question at the end of the question.

2   Abbreviations

The markscheme may make use of the following abbreviations:

(M)   Marks awarded for **Method**

(A)   Marks awarded for an **Answer** or for **Accuracy**

(N)   Marks awarded for correct answers, if **no** working shown: they may not be all the marks for the question. Examiners should only award these marks for correct answers where there is no working.

(R)   Marks awarded for clear **Reasoning**

(AG) **Answer Given** in the question and consequently marks are **not** awarded

Note: In general, it is not possible to award *(M0)(A1)*.

Examiners should use **(d)** to indicate where discretion has been used. It should only be used for decisions on follow through and alternative methods. It must be accompanied by a brief note to explain the decision made

Follow through (ft) marks should be awarded where a correct method has been attempted but error(s) are made in subsequent working which is essentially correct.
 • Penalize the error when it first occurs
 • Accept the incorrect result as the appropriate quantity in all subsequent working
 • If the question becomes much simpler then use discretion to award fewer marks
 • Use **(d)** to indicate where discretion has been used. It should only be used for decisions on follow through and alternative methods. It must be accompanied by a brief note to explain the decision made.

3   Using the Markscheme

(a)   This markscheme presents a particular way in which each question may be worked and how it should be marked. **Alternative methods** have not always been included. Thus, if an answer is wrong then the working must be carefully analysed in order that marks are awarded for a different method in a manner which is consistent with the markscheme. Indicate the awarding of these marks by **(d)**.

Where alternative methods for complete questions or parts of questions are included, they are indicated by **METHOD 1**, **METHOD 2**, *etc*. Other alternative part solutions are indicated by **EITHER….OR**. It should be noted that **G** marks have been removed, and GDC solutions will not be indicated using the **OR** notation as on previous markschemes.

Candidates are expected to show working on this paper, and examiners should not award full marks for just the correct answer. Where it is appropriate to award marks for correct answers with no working, it will be shown on the markscheme using the **N** notation. All examiners will be expected to award marks accordingly in these situations.

(b)   Unless the question specifies otherwise, accept **equivalent forms**. For example: $\dfrac{\sin\theta}{\cos\theta}$ for $\tan\theta$. On the markscheme, these equivalent numerical or algebraic forms will generally be written in brackets after the required answer. Paper setters will indicate the required answer, by allocating full marks at that point. Further working should be ignored, even if it is incorrect. For example: if candidates are asked to factorize a quadratic expression, and they do so correctly, they are awarded full marks. If they then continue and find the roots of the corresponding equation, do not penalize, even if those roots are incorrect, *i.e.* once the correct answer is seen, ignore further working.

(c)   As this is an international examination, all **alternative forms of notation** should be accepted. For example: 1.7, 1·7 , 1,7; different forms of vector notation such as $\vec{u}$, $\overline{u}$, $\underline{u}$; $\tan^{-1}x$ for $\arctan x$.

4   Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

There are two types of accuracy error. Candidates should be penalized **once only IN THE PAPER** for an accuracy error **(AP)**.

Award the marks as usual then write –1**(AP)** against the answer and also on the **front** cover

Rounding errors: only applies to final answers not to intermediate steps.

Level of accuracy: when this is not specified in the question the general rule *unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures* applies.

• If a final correct answer is incorrectly rounded, apply the **AP**
  OR
• If the level of accuracy is not specified in the question, apply the **AP** for answers not given to 3 significant figures. (Please note that this has changed from May 2003).

Note: If there is no working shown, and answers are given to the correct two significant figures, apply the **AP**. However, do **not** accept answers to one significant figure without working.

5   **Graphic Display Calculators**

Many candidates will be obtaining solutions directly from their calculators, often without showing any working. They have been advised that they must use mathematical notation, not calculator commands when explaining what they are doing. Incorrect answers without working will receive no marks. However, if there is written evidence of using a graphic display calculator correctly, method marks may be awarded. Where possible, examples will be provided to guide examiners in awarding these method marks.

Calculator penalties

Candidates are instructed to write the make and model of their calculator on the front cover. Please apply the following penalties where appropriate.

(a)   Illegal calculators

If candidates note that they are using an illegal calculator, please report this on a PRF, and deduct 10% of their overall mark. Note this on the front cover. The most common examples are:

Texas Instruments: TI-89 (plus); TI-92 (plus); TI-Voyage 200
Casio: *fx9970*; *fx2.0* algebra; classpad
HP: 38–95 series

(b)   Calculator box not filled in.

Please apply a calculator penalty *(CP)* of 1 mark if this information is not provided. Note this on the front cover.

Examples

1   **Accuracy**

A question leads to the answer 4.6789….

- 4.68 is the correct 3 s.f. answer.
- 4.7, 4.679 are to the wrong level of accuracy: both should be penalized the first time this type of error occurs.
- 4.67 is incorrectly rounded – penalize on the first occurrence.

Note: All these "incorrect" answers may be assumed to come from 4.6789…, even if that value is not seen, but previous correct working is shown. However, 4.60 is wrong, as is 4.5, 4.8, and these should be penalized as being incorrect answers, not as examples of accuracy errors.

2   Alternative solutions

Question

The points P, Q, R are three markers on level ground, joined by straight paths PQ, QR, PR as shown in the diagram. $QR = 9\,km$, $P\hat{Q}R = 35°$, $P\hat{R}Q = 25°$.
(Note: in the original question, the first part was to find $PR = 5.96$)
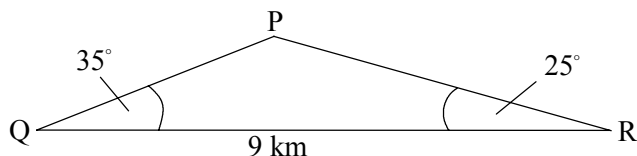

diagram not to scale

(a)   Tom sets out to walk from Q to P at a steady speed of $8\,km\,h^{-1}$. At the same time, Alan sets out to jog from R to P at a steady speed of $a\,km\,h^{-1}$. They reach P at the same time. Calculate the value of a.   *[7 marks]*

(b)   The point S is on [PQ], such that $RS = 2QS$, as shown in the diagram.
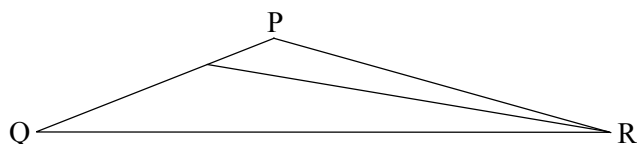
Find the length QS.   *[6 marks]*

Markscheme

(a)   **EITHER**

Sine rule to find PQ

$PQ = \dfrac{9\sin 25}{\sin 120}$ *(M1)(A1)*

$PQ = 4.39\,km$ *(A1)*

OR

Cosine rule: $PQ^2 = 5.96^2 + 9^2 - (2)(5.96)(9)\cos 25$ *(M1)(A1)*

$= 19.29$

$PQ = 4.39\,km$ *(A1)*

THEN

Time for Tom $= \dfrac{4.39}{8}$ *(A1)*

Time for Alan $= \dfrac{5.96}{a}$ *(A1)*

Then $\dfrac{4.39}{8} = \dfrac{5.96}{a}$ *(M1)*

$a = 10.9$ *(A1)*   *(N5)*

[7 marks]

Note that the **THEN** part follows both **EITHER** and **OR** solutions, and this is shown by the alignment.

(b)   **METHOD 1**

$RS^2 = 4QS^2$ *(A1)*

$4QS^2 = QS^2 + 81 - 18 \times QS \times \cos 35$ *(M1)(A1)*

$3QS^2 + 14.74QS - 81 = 0$ (or $3x^2 + 14.74x - 81 = 0$) *(A1)*

$\Rightarrow QS = -8.20$ or $QS = 3.29$ *(A1)*

therefore $QS = 3.29$ *(A1)*   *(N2)*

METHOD 2

$$\frac{QS}{\sin S\hat{R}Q} = \frac{2QS}{\sin 35}$$ *(M1)*

$$\Rightarrow \sin S\hat{R}Q = \frac{1}{2}\sin 35$$ *(A1)*

$S\hat{R}Q = 16.7^\circ$ *(A1)*

Therefore, $Q\hat{S}R = 180 - (35 + 16.7) = 128.3^\circ$ *(A1)*

$$\frac{9}{\sin 128.3} = \frac{QS}{\sin 16.7}\left(= \frac{SR}{\sin 35}\right)$$ *(M1)*

$$QS = \frac{9\sin 16.7}{\sin 128.3}\left(= \frac{9\sin 35}{2\sin 128.3}\right) = 3.29$$ *(A1)*   *(N2)*

If candidates have shown no working award *(N5)* for the correct answer 10.9 in part (a) and *(N2)* for the correct answer 3.29 in part (b).

[6 marks]

3   **Follow through**

Question

Calculate the acute angle between the lines with equations

$$r = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + s \begin{pmatrix} 4 \\ 3 \end{pmatrix} \text{ and } r = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Markscheme

Angle between lines = angle between direction vectors. (May be implied.)   *(A1)*

Direction vectors are $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. (May be implied.)   *(A1)*

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left\| \begin{pmatrix} 4 \\ 3 \end{pmatrix} \right\| \left\| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\| \cos \theta$$ *(M1)*

$$4 \times 1 + 3 \times (-1) = \sqrt{(4^2 + 3^2)} \sqrt{(1^2 + (-1)^2)} \cos \theta$$ *(A1)*

$$\cos \theta = \frac{1}{5\sqrt{2}} \ (= 0.1414 \ldots)$$ *(A1)*

$$\theta = 81.9^\circ \ (1.43 \text{ radians})$$ *(A1)*   *(N3)*

Examples of solutions and marking

| **Solutions** | **Marks allocated** |

1. $\begin{pmatrix} 4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left\| \begin{pmatrix} 4 \\ 3 \end{pmatrix} \right\| \left\| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\| \cos \theta$ *(A1)(A1)* implied
(M1)

$\cos \theta = \dfrac{7}{5\sqrt{2}}$ *(A0)(A1)*

$\theta = 8.13^\circ$ *(A1)***ft**

Total 5 marks

2. $\cos \theta = \dfrac{\begin{pmatrix} 4 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \end{pmatrix}}{\sqrt{17}\sqrt{20}}$ *(A0)(A0)* wrong vectors implied
(M1) for correct method, *(A1)***ft**

$= 0.2169$ *(A1)***ft**
$\theta = 77.5^\circ$ *(A1)***ft**

Total 4 marks

3. $\theta = 81.9^\circ$ *(N3)*

Total 3 marks

Note that this candidate has obtained the correct answer, but not shown any working. The way the markscheme is written means that the first 2 marks may be implied by subsequent correct working, but the other marks are only awarded if the relevant working is seen. Thus award the first 2 implied marks, plus the final mark for the correct answer.

END OF EXAMPLES

1.   (a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$ *(M1)*

$3x - 4y + z = 6$ *(A1)*

[2 marks]

(b)   (i) $1 + 3 \times 2 - 11 = -4$ *(A1)*

$\Rightarrow$ P lies in π_2 *(AG)*

(ii) $\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 13 \end{pmatrix}$ *(M1)(A1)*

$\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ 13 \end{pmatrix}$ *(M1)(A1)*   *(N2)*

Note: Award *(M1)(A0)* if equation given in incorrect form.

[5 marks]

Question 1 continued

(c) **METHOD 1**

$$\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} = \left|\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}\right|\left|\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}\right|\cos\theta$$

$$\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} = -10 \qquad\qquad\qquad \textbf{(A1)}$$

$$\left|\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}\right| = \sqrt{11} \qquad \left|\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}\right| = \sqrt{26} \qquad\qquad \textbf{(A1)}$$

$$\cos\theta = \frac{-10}{\sqrt{11}\ \sqrt{26}} \ (=-0.5913) \qquad\qquad\qquad \textbf{(M1)}$$

$\theta = 2.2035$ radians $(\text{or } \theta = 126.3°)$ $\qquad\qquad\qquad$ **(A1)**

The angle between the planes is
$\pi - 2.2035 = 0.938$ radians $(\text{or } 180° - 126.3° = 53.7°)$ $\qquad$ **(A1)** $\qquad$ **(N2)**

METHOD 2

$$\left|\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}\right| = \left|\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}\right|\left|\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}\right|\sin\theta \qquad\qquad \textbf{(M1)}$$

$$\left|\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}\right| = \left|\begin{pmatrix} 1 \\ 4 \\ 13 \end{pmatrix}\right| = \sqrt{186} \qquad\qquad \textbf{(A1)}$$

$$\left|\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}\right| = \sqrt{11}, \ \left|\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}\right| = \sqrt{26} \qquad\qquad \textbf{(A1)}$$

$$\sin\theta = \frac{\sqrt{186}}{\sqrt{11}\ \sqrt{26}} \ (=0.8064) \qquad\qquad\qquad \textbf{(M1)}$$

$\theta = 0.938$ radians $\left(\text{or } \theta = 53.7°\right)$ $\qquad\qquad\qquad$ **(A1)** $\qquad$ **(N2)**

[5 marks]

Total [12 marks]

2.   (i)   (a)   Probability that Jack wins on his first throw $=\dfrac{2}{3}$ (or 0.667).   *(A1)*

[1 mark]

(b)   Probability that Jill wins on her first throw: $\dfrac{1}{3}\times\dfrac{2}{3}$ *(M1)*

$=\dfrac{2}{9}$ (or 0.222).   *(A1)*   *(N2)*

[2 marks]

(c)   **EITHER**

Probability that Jack wins the game:

$$\left(\dfrac{2}{3}\right)+\left(\dfrac{1}{3}\times\dfrac{1}{3}\times\dfrac{2}{3}\right)+\dots$$ *(M1)*

$$=\dfrac{2}{3}\times\dfrac{1}{1-\dfrac{1}{9}}$$ *(A1)*

$$=\dfrac{3}{4}$$ *(A1)*   *(N2)*

OR

If p is the probability that Jack wins the game then

$$p=\dfrac{2}{3}+\dfrac{1}{3}\times\dfrac{1}{3}\,p,$$ *(M1)*

so that $p=\dfrac{\dfrac{2}{3}}{1-\dfrac{1}{9}}$ *(A1)*

$$=\dfrac{3}{4}$$ *(A1)*   *(N2)*

[3 marks]

continued…

Question 2 continued

(ii) (a) $k\int_0^2 x^2 \mathrm{d}x = 1$ **(M1)**

$\Rightarrow k\left[\dfrac{x^3}{3}\right]_0^2 = \dfrac{k8}{3} = 1$ **(A1)**

$\Rightarrow k = \dfrac{3}{8}$ **(AG)**

[2 marks]

(b) (i) $\mathrm{E}(X) = \dfrac{3}{8}\int_0^2 x.x^2 \mathrm{d}x \quad \left(= \dfrac{3}{8}\left[\dfrac{x^4}{4}\right]_0^2 \right)$ **(M1)**

$= \dfrac{3}{2}$ **(A1)**

(N2)

(ii) The median m must be a number such that

$\dfrac{3}{8}\int_0^m x^2 \mathrm{d}x = \dfrac{1}{2} \quad \left(\text{or } \dfrac{3}{8}\int_m^2 x^2 \mathrm{d}x\right)$ **(M1)(A1)**

$\dfrac{3}{8}\left[\dfrac{x^3}{3}\right]_0^m = \dfrac{3}{8}\left(\dfrac{m^3}{3} - 0\right) = \dfrac{1}{2}$ **(A1)**

$\dfrac{m^3}{8} = \dfrac{1}{2} \Rightarrow m^3 = 4$.

$\Rightarrow m = \sqrt[3]{4} \ (=1.59 \text{ to 3 s.f.})$. **(A1)** **(N4)**

[6 marks]

Total [14 marks]

3.　(i)　(a)　$z = a + b\mathrm{i} \Rightarrow |a + b\mathrm{i}| = |a + (b-3)\mathrm{i}| \Rightarrow a^2 + b^2 = a^2 + (b-3)^2$　　　**(M1)**

$\Rightarrow 6b - 9 = 0$　　　**(A1)**

$\Rightarrow \quad b = \dfrac{3}{2}$　　　**(AG)**

[2 marks]

(b)　(i)
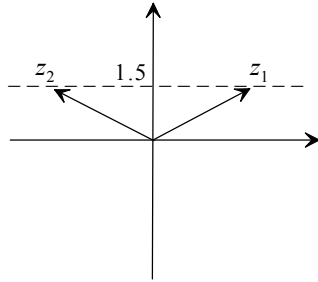
| **Note:** | Award *(A1)* for the sketch and *(A1)* for the labels. |

(A1)
(A1)

(ii)　$\arg z_1 = \arcsin\left(\dfrac{3}{2} \div 3\right)$　　　**(A1)**

$= \dfrac{\pi}{6}$　　　**(AG)**

(iii)　$\arg z_2 = \pi - \arg z_1 = \dfrac{5}{6}\pi$　　　**(A1)**

[4 marks]

(c)　$k \arg z_1 + \arg z_2 - \arg 2\mathrm{i} = \pi$　　　**(M1)(A1)**

$\dfrac{\pi}{6}k + \dfrac{5}{6}\pi - \dfrac{\pi}{2} = \pi$　　　**(A1)**

$k = 4$ (accept $k = 4 \pm 12n$)　　　**(A1)**

[4 marks]

Question 3 continued

(ii) **METHOD 1**

$$\ln x^2 + \ln \frac{x^2}{y} + \ln \frac{x^2}{y^2} + \ln \frac{x^2}{y^3} + \ldots$$

$$= \ln x^2 + (\ln x^2 - \ln y) + (\ln x^2 - 2\ln y) + (\ln x^2 - 3\ln y) + \ldots \qquad \textbf{\textit{(M1)}}$$

$$S_{35} = \frac{n}{2}\left(2u_1 + (n-1)d\right) = \frac{35}{2}(2\ln x^2 - 34\ln y) = 35\ln x^2 - 595\ln y \qquad \textbf{\textit{(M1)(A1)}}$$

$$= \ln x^{70} - \ln y^{595}$$

$$= \ln \frac{x^{70}}{y^{595}} \text{ (Accept } m = 70, n = 595) \qquad \textbf{\textit{(A1)(A1)}} \qquad \textbf{\textit{(N2)}}$$

METHOD 2

$$\ln x^2 + \ln \frac{x^2}{y} + \ln \frac{x^2}{y^2} + \ln \frac{x^2}{y^3} + \ldots = \ln \frac{x^2 x^2 \ldots x^2}{1y \ldots y^{34}} \qquad \textbf{\textit{(M1)(A1)}}$$

In the denominator, the sum of the powers of y is $(0+34)\dfrac{35}{2} = 595$ $\qquad$ $\textbf{\textit{(A1)}}$

The required expression is

$\ln \dfrac{x^{70}}{y^{595}}$ (Accept $m = 70, n = 595$) $\qquad\qquad\qquad\qquad$ $\textbf{\textit{(A1)(A1)}}$ $\qquad$ $\textbf{\textit{(N2)}}$

[5 marks]

Total [15 marks]

4.   (a) $\dfrac{\mathrm{d}T}{\mathrm{d}t} = k(T-22) \Rightarrow \displaystyle\int \dfrac{\mathrm{d}T}{T-22} = \int k\,\mathrm{d}t$ *(M1)*

$\ln|T-22| = kt + c \;\big(\text{accept } \ln(T-22)\big)$ *(A1)*

$T - 22 = A\mathrm{e}^{kt}$ *(A1)*

$T = A\mathrm{e}^{kt} + 22$ *(AG)*

[3 marks]

(b)   (i)   When $t = 0,\, T = 100 \Rightarrow 100 = A\mathrm{e}^{0} + 22$ *(M1)*

$A = 78$ *(A1)*

When $t = 15,\, T = 70 \Rightarrow 70 = 78\mathrm{e}^{15k} + 22$ *(M1)*

$\Rightarrow k = \dfrac{1}{15}\ln\dfrac{48}{78} \;\big(= -0.0324\big)$ *(A1)*

(ii) $40 = 78\mathrm{e}^{-0.0324t} + 22$ *(A1)*

$\mathrm{e}^{-0.0324t} = \dfrac{18}{78}$ *(A1)*

$t = -\dfrac{1}{0.0324}\ln\dfrac{18}{78} \;\big(= 45.3\big)$ *(A1)*

[7 marks]

Total [10 marks]

5. (a) $\cos(A+B)=\cos A\cos B-\sin A\sin B$, $\cos(A-B)=\cos A\cos B+\sin A\sin B$ **(M1)(A1)**

Hence $\cos(A+B)+\cos(A-B)=2\cos A\cos B$ **(AG)**

[2 marks]

(b) (i) $T_1(x)=\cos(\arccos x)$ **(M1)**

$\quad\quad = x$ **(A1)**

(ii) $T_2(x)=\cos(2\arccos x)$ **(A1)**

$\quad\quad = 2\cos^2(\arccos x)-1$ **(A2)**

$\quad\quad = 2x^2-1$ **(AG)**

[5 marks]

(c) (i) $T_{n+1}(x)+T_{n-1}(x)=\cos\big[(n+1)\arccos x\big]+\cos\big[(n-1)\arccos x\big]$ **(A1)**

Using part (a) with $A=n\arccos x,\ B=\arccos x$ **(M1)**

$T_{n+1}(x)+T_{n-1}(x)=2\cos(n\arccos x)\cos(\arccos x)$ **(A1)**

$\quad\quad = 2x\cos(n\arccos x)$ **(A1)**

$\quad\quad = 2xT_n(x)$ **(AG)**

(ii) Let P_n be the statement: $T_n(x)$ is a polynomial of degree, $n\in\mathbb{Z}^+$

$T_1(x)=x$, a polynomial of degree one.   **(A1)**

So P_1 is true.

$T_2(x)=2x^2-1,$ is a polynomial of degree two.   **(A1)**

So P_2 is true.

Assume that P_k is true.   **(M1)**

From part (c)(i), $T_{k+1}(x)=2xT_k(x)-T_{k-1}(x)$ **(M1)**

Assume P_{k-1} is true as well.   **(M1)**

$T_k(x)$ has degree k

$\Rightarrow 2xT_k(x)$ has degree $(k+1)$ **(A1)**

and as $T_{k-1}(x)$ has degree $(k-1)$

$\Rightarrow T_{k+1}(x)$ has degree $(k+1)$ **(A1)**

By the principle of mathematical induction, P_n is true for all positive integers *n*.   **(R1)**

Notes: These arguments may be in a different order.

There is a maximum of 6 marks in part (ii) for candidates who do not consider a two stage process.

[12 marks]

Total [19 marks]

6. **Note:** In this question do not penalize accuracy for more than 3 s.f.

(i) (a) Unbiased estimate of mean $= \dfrac{331.8}{10} = 33.18$ (accept 33.2)   **(A1)**

Unbiased estimate of variance $= 3.22$ (3 s.f.)   **(A2)**

Note: Award **(A1)** for 3.21.

[3 marks]

(b) 90 % confidence limits are $\dfrac{331.8}{10} \pm 1.833\sqrt{\dfrac{3.215}{10}}$ **(M1)(A1)**

giving [32.1, 34.2]   **(A1)**   **(N3)**

[3 marks]

(ii) (a) The null hypothesis is:
H_0 : There is no association between classification in exams and gender.   **(A1)**

[1 mark]

(b) Under H_0 , the expected frequencies are: $\dfrac{54 \times 113}{193} = 31.62$ *etc*.

The table of expected frequencies is

	Distinction	Pass	Fail
Male	31.6	68.5	12.9
Female	22.4	48.5	9.12

(A4)

Note: Award **(A4)** for all 6 correct, **(A2)** for 5, **(A0)** otherwise.

[4 marks]

(c) $\chi^2 = \dfrac{(26-31.6)^2}{31.6} + \ldots = 4.03$ (accept 4.04)   **(M1)(A1)**   **(N2)**

[2 marks]

(d) Degrees of freedom $= 2$ **(A1)**
Critical value $= 5.991$ (or p-value $= 0.133$)   **(A1)**
There is insufficient evidence, at the 5 % level, to conclude that there is any
association between classification and gender (or equivalent statement,
eg accept H_0).   **(R1)**

[3 marks]

Question 6 continued

(iii) (a) $H_0 : \mu_A = \mu_B$; (Accept equivalent in words.)   *(A1)*

[1 mark]

(b)   Pooled estimate $= \dfrac{29 \times 4.85 + 31 \times 5.38}{60} = 5.12$ *(M1)(A1)*   *(N2)*

[2 marks]

(c)   **METHOD 1**

t-statistic $= \dfrac{65.51 - 64.27}{\sqrt{5.12\left(\dfrac{1}{30} + \dfrac{1}{32}\right)}} = 2.16$ *(M1)(A1)*   *(N2)*

(i)   5 % critical value = 2.000 so significant at 5 % level   *(A1)*

(ii)   1 % critical value = 2.660 so not significant at 1 % level.   *(A1)*

METHOD 2

p-value = 0.035(1)   *(A2)*

(i)   significant at 5 % level   *(A1)*
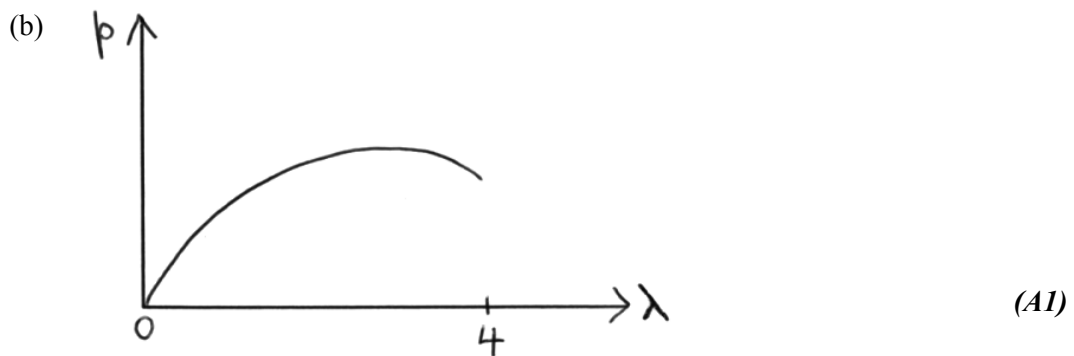
(ii)   not significant at 1 % level   *(A1)*

[4 marks]

(iv) (a) $p = \lambda e^{-\lambda} + \dfrac{\lambda^2}{2} e^{-\lambda}$ *(A1)*

[1 mark]

(b)

(A1)

| **Note:** | Award *(A1)* for a maximum point in [0, 4]; sketch need not be accurate. |

[1 mark]

continued…

Question 6 (iv) continued

(c) $\dfrac{\mathrm{d}p}{\mathrm{d}\lambda} = \mathrm{e}^{-\lambda} + \lambda(-\mathrm{e}^{-\lambda}) + \mathrm{e}^{-\lambda} \times \lambda + \dfrac{\lambda^2}{2}(-\mathrm{e}^{-\lambda})$ *(M1)(A1)*

$\quad\quad = \mathrm{e}^{-\lambda}\left(1 - \dfrac{\lambda^2}{2}\right)$ *(A1)*

$\quad p_{\max}$ when $\dfrac{\mathrm{d}p}{\mathrm{d}\lambda} = 0 \Rightarrow \left(1 - \dfrac{\lambda^2}{2}\right) = 0$ *(M1)*

$\quad \Rightarrow \lambda^2 = 2$
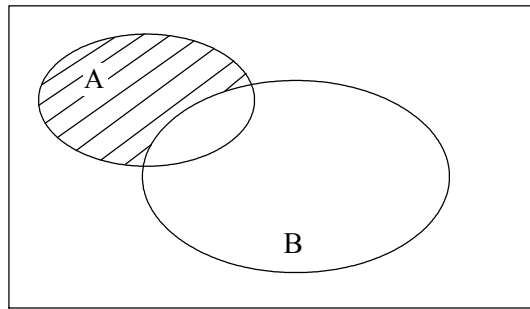$\quad \Rightarrow \lambda = \sqrt{2}$ (do **not** accept 1.41)   *(A1)*

Note:	If no working shown, award *(A2)* for an answer of 1.41 obviously obtained from a GDC.

[5 marks]

Total [30 marks]

7. (i) (a)

The shaded area denotes $A - B$ and $A \cap B'$ *(A1)*
confirming that $A - B = A \cap B'$.   *(AG)*

[1 mark]

(b) $A - (B \cup C) = A \cap (B \cup C)'$ *(M1)*
$$= A \cap (B' \cap C')$$
$$= A \cap B' \cap C'$$ *(A1)*
$(A - B) \cap (A - C) = (A \cap B') \cap (A \cap C')$ *(M1)*
$$= A \cap B' \cap A \cap C'$$
$$= A \cap A \cap B' \cap C'$$
$$= A \cap B' \cap C'$$ *(A1)*

[4 marks]

(ii) (a) Reflexive: $7^a \equiv 7^a$ (modulo 10) so aRa *(A1)*
Symmetric: $7^a \equiv 7^b$ (modulo 10) $\Rightarrow 7^b \equiv 7^a$ (modulo 10) so $aRb \Rightarrow bRa$ *(A1)*

Transitive: Let $7^a \equiv 7^b$ (modulo 10) and $7^b \equiv 7^c$ (modulo 10)   *(M1)*
Then, $7^a = 7^b + 10\lambda$ and $7^b = 7^c + 10\mu$
so $7^a = 7^c + 10(\lambda + \mu)$ so aRb and $bRc \Rightarrow aRc$ *(A1)*

[4 marks]

(b)   We note that $7^0 = 1, 7^1 = 7, 7^2 = 49, 7^3 = 343, 7^4 = 2401$
The equivalence classes are therefore
       0, 4, 8, …   *(A1)*
       1, 5, 9, …   *(A1)*
       2, 6, 10, …   *(A1)*
       3, 7, 11, …   *(A1)*

[4 marks]

(c) 7^{503} (modulo 10) $\equiv 7^3$ (modulo 10) $= 3$.   *(A1)*

[1 mark]

continued…

Question 7 continued

(iii)   (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$ *(A1)(A1)(A1)(A1)*

[4 marks]

(b)   (i)   Negative determinants $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has order 2   *(A1)*

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$ *(A1)*

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ has order 3   *(A1)*

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$ *(A1)*

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ has order 3   *(A1)*

(ii)   Subgroup is $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ *(A1)*

[6 marks]

(iv)   Consider $a*b$. This cannot be a or b since $a*b=a \Rightarrow b=e$ which is not the   *(M1)*
case and similarly for b. So $a*b=$ either e or c.   *(R1)*
If $a*b=e$, then a,b form an inverse pair so $b*a=e$.   *(R1)*
Suppose $a*b=c$. Consider $b*a$. As before, this cannot equal a or b and it
cannot equal e either because that would imply that $a*b=e$ which it is not.   *(R1)*
It follows that $b*a=c$.   *(R1)*
Thus in both cases, $a*b=b*a$.   *(R1)*

[6 marks]

Total [30 marks]

8.   (i)   The characteristic equation is $r^2 - 3r - 28 = 0$ **(M1)**

                              so that $r = -4$, or $r = 7$ **(A1)**

         The general solution is $x_n = A(-4)^n + B(7)^n$ **(A1)**

         Setting $n = 0, 1$ we obtain

$$A + B = 7 , \quad -4A + 7B = -6$$ **(M1)**

         $A = 5, B = 2$

         So $x_n = 5(-4)^n + 2(7)^n$ (Accept $A = 5, B = 2$.)   **(A1)**

                                                                                [5 marks]

   (ii) (a) (i)   A bipartite graph is a graph whose vertices can be divided into
                    two sets and in which edges always connect a vertex from one set
                    to a vertex from the other set.   **(A2)**

         | **Note:**   Accept equivalent definitions, *e.g.* chromatic number = 2. |

              (ii) An isomorphism between two graphs M and N is a one-to-one
                    correspondence between vertices which maps the adjacency   **(A1)**
                    matrix of M onto the adjacency matrix of N.   **(A1)**

         | **Note:**   This definition can be simplified when M and N are simple graphs. |

                                                                                [4 marks]

         (b) Let M contain a cycle $(x_1, x_2); (x_2, x_3); \ldots (x_n, x_1)$ **(M1)**

              Let Φ denote an isomorphism between M and N.

              Consider $\Phi(x_1, x_2); \Phi(x_2, x_3); \ldots \Phi(x_n, x_1)$ **(M1)**

              The adjacency property shows that this is a cycle, proving the result.   **(R1)**

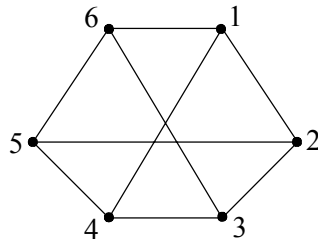 | **Note:**   Accept less formal proofs in which the above statements are made verbally rather than mathematically. |

                                                                                [3 marks]

                                                                                continued...
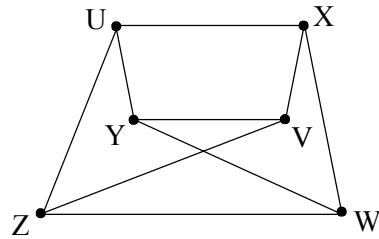
Question 8 continued

(c)　(i)

The graph is bipartite   *(A1)*
because by putting $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$, the conditions
in (ii) (a) (i) are satisfied.   *(R1)*

(ii)

The graphs are isomorphic   *(A1)*
Consider the mapping
$1 \rightarrow U, 2 \rightarrow X, 3 \rightarrow V, 4 \rightarrow Y, 5 \rightarrow W, 6 \rightarrow Z$ *(M1)*
Since adjacency is preserved, the graphs are isomorphic.   *(R1)*

Note:　　Other isomorphisms exist – please check any answer carefully.

(iii)　The graphs are not isomorphic   *(A1)*
because

EITHER

J contains a cycle or order 3 but *H* does not.   *(R1)*

OR

H is bipartite but *J* is not   *(R1)*
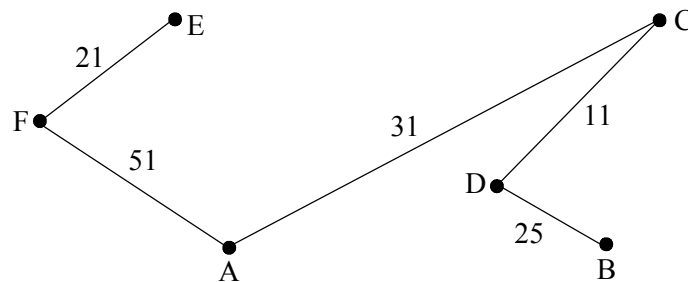
[*7 marks*]

continued...

(Question 8 continued)

(iii) (a)   Let *S* be a set of vertices and *T* be a set of edges. The algorithm is
            organised as follows:

Choice	Edge	Weight	*T*	*S*	
1	CD	11	{CD}	{C,D}	*(M1)(A1)*
2	EF	21	{CD, EF}	{C,D,E,F}	
3	BD	25	{CD, EF, BD}	{C,D,E,F,B}	
4	AC	31	{CD, EF, BD, AC}	{A,B,C,D,E,F}	*(A1)*
5	AF	51	{CD, EF, BD, AC, AF,}	{A,B,C,D,E,F}	*(A1)*

> **Note:**   Accept any solution which indicates the order in which the edges
> are selected.

[4 marks]

(b)   The minimal spanning tree is (weights not needed)

(A1)

Total weight = 139   *(A1)*

[2 marks]

(iv) (a)   Every non-empty set of positive integers (or subset of $\mathbb{Z}^+$) contains
            a least element.   *(A1)(A1)*

> **Note:**   Award *(A1)* for "non-empty" set and *(A1)* for "positive integers or ($\mathbb{Z}^+$)".

[2 marks]

(b)   Assume that the statement is not true, *i.e.* for some integers a, b, $na < b$
       for every positive integer *n*. Let

$$S = \{b - na \mid n \text{ is a positive integer}\} \qquad (M1)$$

Then *S* consists entirely of positive integers. By the well-ordering
principle, *S* has a least element, say $b - ma$.   *(M1)*
But $b - (m+1)a < b - ma$ showing that $b - ma$ is not the smallest element
in *S*, resulting in a contradiction. Hence the given statement is true.   *(R1)*

[3 marks]

Total [30 marks]

9. | **Note:**   In this question do not penalize accuracy for more than 3 s.f. |

(i)   **METHOD 1**
We note first that $f(0) = 0$.   *(A1)*

$$f'(x) = \frac{\cos x}{1 + \sin x};$$ *(M1)*

$$f'(0) = 1$$ *(A1)*

$$f''(x) = \frac{-\sin x(1 + \sin x) - \cos^2 x}{(1 + \sin x)^2} = \frac{-1}{(1 + \sin x)}, f''(0) = -1$$ *(A1)*

$$f'''(x) = \frac{\cos x}{(1 + \sin x)^2}, f'''(0) = 1$$ *(A1)*

$$f''''(x) = \frac{-\sin x(1 + \sin x)^2 - 2(1 + \sin x)\cos^2 x}{(1 + \sin x)^4}, f^{(4)}(0) = -2$$ *(A1)*

| **Note:**   Award full marks for numerical derivatives obtained from a GDC. |

The series is $f(x) = x - \dfrac{x^2}{2!} + \dfrac{x^3}{3!} - \dfrac{2x^4}{4!} + ...$ *(M1)(A1)*

METHOD 2

$$\ln(1 + \sin x) = \sin x - \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3} - \frac{\sin^4 x}{4} + ...$$ *(M1)(A1)*

$$= x - \frac{x^3}{6} + ... - \frac{1}{2}\left(x - \frac{x^3}{6} + ...\right)^2 + \frac{1}{3}(x + ...)^3 - \frac{1}{4}(x + ...)^4$$

                                                                                  (M1)(A1)

$$= x - \frac{x^3}{6} + ... - \frac{1}{2}x^2 + \frac{1}{6}x^4 + ... + \frac{1}{3}x^3 - \frac{1}{4}x^4 + ...$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{1}{12}x^4 + ...$$ *(A1)(A1)(A1)(A1)*

                                                                                  [8 marks]

(ii)   (a)   Using the trapezium rule,
$$\text{Int} \cong \frac{0.25}{2}(\sin 0.5^2 + 2\sin 0.75^2 + \sin 1) = 0.269435$$ *(M1)(A1)*   *(N2)*

                                                                                  [2 marks]

(b)   To find an error bound, consider
$$f'(x) = 2x\cos(x^2)$$ *(A1)*
$$f''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$$ *(A1)*
Graphing $|f''(x)|$, it is clear that its maximum absolute value is when
$x = 1$, or from graph maximum absolute value is 2.285279327 .   *(A1)*

Error bound $= |2\cos 1 - 4\sin 1| \times \dfrac{0.5 \times 0.25^2}{12} = 0.00595$ *(M1)(A1)*

                                                                                  [5 marks]
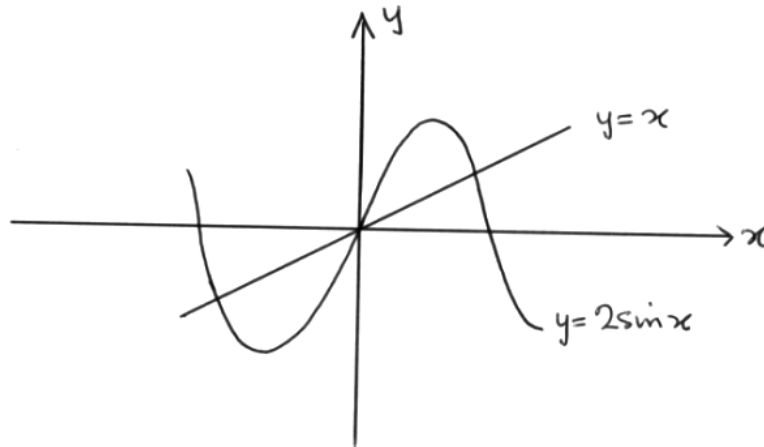
(c)   Using a calculator, the value of the integral is 0.268787 .   *(A1)*
Actual error $= 0.269435 - 0.268787 = 0.000648 <$ error bound.   *(A1)*

                                                                                  [2 marks]

Question 9 continued

(iii) (a)

The sketch indicates 3 real roots.   *(A1)*

Note: Other sketches are possible, *e.g.* $y = \sin x$, $y = \dfrac{1}{2}x$ and $y = 2\sin x - x$.

[2 marks]

(b)   Using the GDC, $\alpha = 1.8954942670$.   *(A1)*

[1 mark]

(c)   The Newton-Raphson iteration is
$$x \rightarrow x - \frac{(2\sin x - x)}{(2\cos x - 1)}$$ *(M1)(A1)*
Successive iterates are
$$x_0 = 2$$
$$x_1 = 1.900995594$$ *(A1)*
$$x_2 = 1.895511645$$ *(A1)*

[4 marks]

(d)   Substituting, we obtain
$$0.005501327 = k \times 0.104505733^N$$ *(A1)*
$$0.000017378 = k \times 0.005501327^N$$ *(A1)*
Dividing,
$$316.5684774 = 18.99645904^N$$ *(A1)*
So that
$$N = \frac{\ln 316.5684774}{\ln 18.99645904} = 1.96$$ *(A1)*
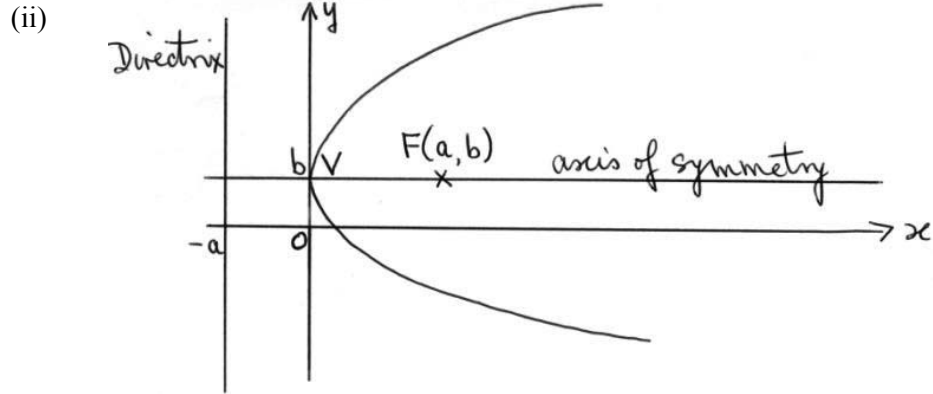So take $N = 2$.   *(R1)*
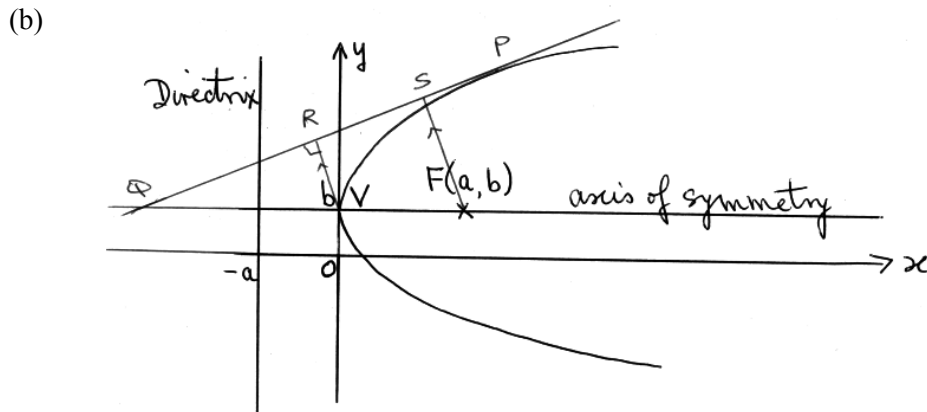The order of convergence is 2.   *(R1)*

[6 marks]

Total [30 marks]

10. (i) (a) (i) Let (x, y) be on C. Then, using the focus-directrix definition,
$$(x-a)^2 + (y-b)^2 = (x+a)^2 \qquad \textbf{(M1)}$$
$$(y-b)^2 = (x+a)^2 - (x-a)^2 = 4ax \qquad \textbf{(A1)}$$

(ii)

(A1)(A1)(A1)(A1)

> **Note:** Award **(A1)** for graph, **(A1)** for vertex, **(A1)** for axis of symmetry and **(A1)** for focus and directrix.

[6 marks]

(b)

$$x = \frac{3a}{2} \Rightarrow (y-b)^2 = 6a^2 \Rightarrow y = a\sqrt{6} + b$$

So P is the point $\left(\dfrac{3a}{2}, a\sqrt{6}+b\right)$. **(A1)**

We now need the gradient of the tangent:
$$2(y-b)\frac{dy}{dx} = 4a$$

At P, $\dfrac{dy}{dx} = \dfrac{2}{\sqrt{6}}$ **(M1)(A1)**

Equation of PQ is
$$y - a\sqrt{6} - b = \frac{2}{\sqrt{6}}\left(x - \frac{3a}{2}\right) \qquad \text{(I)} \qquad \textbf{(M1)(A1)}$$

To find the coordinates of Q, put $y = b$.

continued…

Question 10 (i) (b) continued

This gives $x = -\dfrac{3a}{2}$ so Q is $\left(-\dfrac{3a}{2}, b\right)$.   *(A1)*

The gradient of RV is $-\dfrac{\sqrt{6}}{2}$.   *(A1)*

The equation of RV is

$$y - b = -\frac{\sqrt{6}}{2}x \qquad\qquad \text{(II)}$$ *(A1)*

The coordinates of R are found by solving (I) and (II) simultaneously.

It follows that the *x*-coordinate of R is $-\dfrac{3a}{5}$.   *(M1)(A1)*

$$\frac{PR}{RQ} = \frac{\left(\dfrac{3a}{2} + \dfrac{3a}{5}\right)}{\left(-\dfrac{3a}{5} + \dfrac{3a}{2}\right)} = \frac{7}{3}$$ *(M1)(A1)*
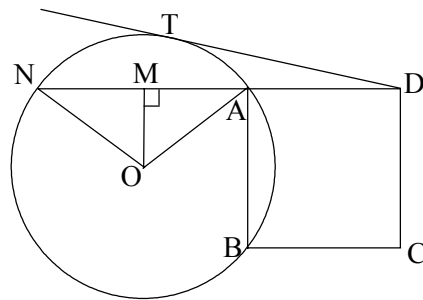
[12 marks]

(c) The equation of FS is

$$y - b = -\frac{\sqrt{6}}{2}(x - a) \qquad\qquad \text{(III)}$$ *(M1)(A1)*

The coordinates of S are found by solving (I) and (III) simultaneously.

This gives $S\left(0, \dfrac{a\sqrt{6}}{2} + b\right)$.   *(A1)(A1)*

[4 marks]

(ii)

(A2)

$$AM = \sqrt{r^2 - \frac{a^2}{4}} \text{ so } AN = 2\sqrt{r^2 - \frac{a^2}{4}}$$ *(M1)(A1)*

Use the result $DT^2 = DA \times DN$.   *(M1)*

$$(2a)^2 = a \times \left(a + 2\sqrt{r^2 - \frac{a^2}{4}}\right)$$ *(A1)*

leading to $r = \dfrac{\sqrt{10}a}{2}$.   *(M1)(A1)*

[8 marks]

Total [30 marks]