INTERNATIONAL BACCALAUREATE
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N02/510/H(2)M+

# MARKSCHEME

## November 2002

## MATHEMATICS

## Higher Level

## Paper 2

25 pages

# Paper 2 Markscheme

## Instructions to Examiners

**1 Method of marking**

(a) All marking must be done using a **red** pen.

(b) Marks should be noted on candidates' scripts as in the markscheme:

- show the breakdown of individual marks using the abbreviations ***(M1)***, ***(A2)*** *etc*.
- write down each part mark total, indicated on the markscheme (for example, ***[3 marks]*** ) – it is suggested that this be written at the end of each part, and underlined;
- write down and circle the total for each question at the end of the question.

**2 Abbreviations**

The markscheme may make use of the following abbreviations:

***M*** Marks awarded for **Method**

***A*** Marks awarded for an **Answer** or for **Accuracy**

***G*** Marks awarded for correct solutions, generally obtained from a **Graphic Display Calculator**, irrespective of working shown

***C*** Marks awarded for **Correct** statements

***R*** Marks awarded for clear **Reasoning**

***AG*** **Answer Given** in the question and consequently marks are **not** awarded

**3 Follow Through (ft) Marks**

Questions in this paper were constructed to enable a candidate to:

- show, step by step, what he or she knows and is able to do;
- use an answer obtained in one part of a question to obtain answers in the later parts of a question.

Thus errors made at any step of the solution can affect all working that follows. Furthermore, errors made early in the solution can affect more steps or parts of the solution than similar errors made later.

To limit the severity of the penalty for errors made at any step of a solution, **follow through (ft)** marks should be awarded. The procedures for awarding these marks require that all examiners:

(i) penalise an error when it **first occurs**;

(ii) **accept the incorrect answer** as the appropriate value or quantity to be used in all subsequent parts of the question;

(iii) award ***M*** marks for a correct method, and ***A*(ft)** marks if the subsequent working contains no further errors.

**Follow through** procedures may be applied repeatedly throughout the same problem.

The errors made by a candidate may be: arithmetical errors; errors in algebraic manipulation; errors in geometrical representation; use of an incorrect formula; errors in conceptual understanding.

The following illustrates a use of the **follow through** procedure:

| Markscheme | | Candidate's Script | Marking | |
|---|---|---|---|---|
| \$ 600 × 1.02 | ***M1*** | Amount earned = \$ 600 × 1.02 | ✓ | ***M1*** |
| = \$ 612 | ***A1*** | = \$602 | × | ***A0*** |
| \$ (306 × 1.02) + (306 × 1.04) | ***M1*** | Amount = 301 × 1.02 + 301 × 1.04 | ✓ | ***M1*** |
| = \$ 630.36 | ***A1*** | = \$ 620.06 | ✓ | ***A1*(ft)** |

**Note that** the candidate made an arithmetical error at line 2; the candidate used a correct method at lines 3, 4; the candidate's working at lines 3, 4 is correct.

However, if a question is transformed by an error into a **different, much simpler question** then:

(i) **fewer** marks should be awarded at the discretion of the Examiner;

(ii) marks awarded should be followed by '**(d)**' (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(iii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

**4 Using the Markscheme**

(a) This markscheme presents a particular way in which each question may be worked and how it should be marked. **Alternative methods** have not always been included. Thus, if an answer is wrong then the working must be carefully analysed in order that marks are awarded for a different method in a manner which is consistent with the markscheme.

In this case:

(i) a mark should be awarded followed by '**(d)**' (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(ii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

Alternative solutions are indicated by **OR**. Where these are accompanied by ***G*** marks, they usually signify that the answer is acceptable from a graphic display calculator without showing working. For example:

Mean = 7906/134 ***(M1)***
= 59 ***(A1)***

**OR**

Mean = 59 ***(G2)***

(b) Unless the question specifies otherwise, accept **equivalent forms**. For example: $\frac{\sin\theta}{\cos\theta}$ for $\tan\theta$

These equivalent numerical or algebraic forms may be written in brackets after the required answer.

(c) As this is an international examination, all **alternative forms of notation** should be accepted. For example: 1.7 , 1·7 , 1,7 ; different forms of vector notation such as $\vec{u}$ , $\bar{u}$ , $\underline{u}$ ; $\tan^{-1} x$ for arctan $x$.

**5 Accuracy of Answers**

There are two types of accuracy errors, incorrect level of accuracy, and rounding errors. Unless the level of accuracy is specified in the question, candidates should be penalized **once only IN THE PAPER** for any accuracy error **(AP)**. This could be an incorrect level of accuracy, or a rounding error. Hence, on the **first** occasion in the paper when a correct answer is given to the wrong degree of accuracy, or rounded incorrectly, maximum marks are **not** awarded, but on **all subsequent occasions** when accuracy errors occur, then maximum marks **are** awarded.

There are also situations (particularly in some of the options) where giving an answer to more than 3 significant figures is acceptable. This will be noted in the markscheme.

(a) **Level of accuracy**

(i) In the case when the accuracy of the answer is **specified in the question** (for example: "find the size of angle $A$ to the nearest degree") the maximum mark is awarded **only if** the correct answer is given to the accuracy required.

(ii) When the accuracy is **not** specified in the question, then the general rule applies:

*Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.*

(b) **Rounding errors**

Rounding errors should only be penalized at the **final answer** stage. This does **not** apply to intermediate answers, only those asked for as part of a question. Premature rounding which leads to incorrect answers should only be penalized at the answer stage.

Incorrect answers are wrong, and should not be considered under (a) or (b).

**Examples**

A question leads to the answer 4.6789….

- 4.68 is the correct 3 s.f. answer.
- 4.7, 4.679 are to the wrong level of accuracy, and should be penalised the first time this type of error occurs.
- 4.67 is incorrectly rounded – penalise on the first occurrence.

**Note**: All these "incorrect" answers may be assumed to come from 4.6789..., even if that value is not seen, but previous correct working is shown. However, 4.60 is wrong, as is 4.5, 4.8, and these should be penalised as being incorrect answers, not as examples of accuracy errors.

| Markscheme | | Candidate's Script (A) | Marking |
|---|---|---|---|
| (a) $a = 2.31 \times 3.43$<br>$= 7.9233 = 7.92$ (3 s.f.) | ***M1***<br>***A1*** | (a) $a = 2.31 \times 3.43$<br>$= 7.9233 = 7.92$ | ***M1***<br>***A1*** |
| (b) $2a = 2 \times 7.9233$<br>$= 15.8466 = 15.8$ (3 s.f.) | ***A1***<br>***A1*** | (b) $2a = 2 \times 7.29 = 14.58$<br>$= 14.5$ | ***A0***<br>***A0*(AP)** |
| | | Total | *2 marks* |

**Notes:** Award ***A1*** for either the exact answer 7.9233 or the 3 s.f. answer 7.92.

In line 3, Candidate A has incorrectly transcribed the answer for part (a), but then performs the calculation correctly, and would normally gain the follow through marks. However, the final answer is incorrectly rounded, and the **AP** applies.

| Candidate's Script (B) | Marking | | Candidate's Script (C) | Marking |
|---|---|---|---|---|
| (a) $a = 2.31 \times 3.43 = 7.9233$<br>$= 7.92$ | ***M1***<br>***A1*** | | (a) $a = 2.31 \times 3.43 = 7.9233$<br>$= 7.93$ | ***M1***<br>***A0*(AP)** |
| (b) $2a = 2 \times 7.9233$<br>$= 15.8466 = 15.85$ | ***A1***<br>***A0*(AP)** | | (b) $2a = 2 \times 7.93$<br>$= 15.86 \quad = 15.8$ | ***A1*(ft)**<br>***A1*(ft)** |
| Total | *3 marks* | | Total | *3 marks* |

**Notes:** Candidate B has given the answer to part (b) to the wrong level of accuracy, **AP** applies.

Candidate C has incorrectly rounded the answers to both parts (a) and (b), is penalised (**AP**) on the first occurrence (line 2), and awarded follow through marks for part (b).

| Candidate's Script (D) | Marking | | Candidate's Script (E) | Marking |
|---|---|---|---|---|
| (a) $a = 2.31 \times 3.43$<br>$= 7.923 = 7.9$ | ***M1***<br>***A0*(AP)** | | (a) $a = 2.31 \times 3.43 = 7.923$<br>$= 7.93$ | ***M1***<br>***A0*(AP)** |
| (b) $2a = 2 \times 7.923$<br>$= 19.446 = 19.5$ | ***A1*(ft)**<br>***A0*** | | (b) $2a = 2 \times 7.93$<br>$= 15.86$ | ***A1*(ft)**<br>***A1*(ft)** |
| Total | *2 marks* | | Total | *3 marks* |

**Notes:** Candidate D has given the answer to part (a) to the wrong level of accuracy, and therefore loses 1 mark (**AP**). The answer to part (b) is wrong.

Candidate E has incorrectly rounded the answer to part (a), therefore loses 1 mark (**AP**), is awarded follow through marks for part (b), and does **not** lose a mark for the wrong level of accuracy.

## 6 Graphic Display Calculators

Many candidates will be obtaining solutions directly from their calculators, often without showing any working. They have been advised that they must use mathematical notation, not calculator commands when explaining what they are doing. Incorrect answers without working will receive no marks. However, if there is written evidence of using a graphic display calculator correctly, method marks may be awarded. Where possible, examples will be provided to guide examiners in awarding these method marks.

**1.** (i) (a) $u_2 = 3(2) - 2(1)$ $u_3 = 3(4) - 2(2)$ $u_4 = 3(8) - 2(4)$

$u_2 = 4$ $u_3 = 8$ $u_4 = 16$ ***(A1)(A1)(A1)***

***[3 marks]***

(b) (i) Conjecture is $u_n = 2^n$ ***(C1)***

(ii) $3(u_n) - 2(u_{n-1}) = 3(2^n) - 2(2^{n-1})$ ***(M1)***

$= 3(2^n) - 2^n$

$= 2(2^n)$

$= 2^{n+1}$ ***(A1)***

$= u_{n+1}$ ***(AG)***

***[3 marks]***

(ii) (a) $\boldsymbol{M}^2 = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$ ***(A1)***

$\boldsymbol{M}^3 = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$ ***(A1)***

$\boldsymbol{M}^4 = \begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix}$ ***(A1)***

***[3 marks]***

(b) $\boldsymbol{M}^n = \begin{pmatrix} n+1 & -n \\ n & 1-n \end{pmatrix}$ for $n \in \mathbb{Z}^+$ ***(C1)***

Let P($n$) be the statement $\boldsymbol{M}^n = \begin{pmatrix} n+1 & -n \\ n & 1-n \end{pmatrix}$

$\boldsymbol{M}^1 = \begin{pmatrix} 1+1 & -1 \\ 1 & 1-1 \end{pmatrix}$ ***(M1)***

$= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ ***(AG)***

Therefore, P(1) is true.

Assume P($k$) is true, *i.e.* $\boldsymbol{M}^k = \begin{pmatrix} k+1 & -k \\ k & 1-k \end{pmatrix}$ ***(M1)***

$\boldsymbol{M}^{k+1} = \boldsymbol{M}^k \cdot \boldsymbol{M}$

$= \begin{pmatrix} k+1 & -k \\ k & 1-k \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ ***(M1)***

$= \begin{pmatrix} 2k+2-k & -k-1 \\ 2k+1-k & -k \end{pmatrix}$ ***(A1)***

$= \begin{pmatrix} (k+1)+1 & -(k+1) \\ k+1 & 1-(k+1) \end{pmatrix}$ ***(A1)***

Therefore, the truth of P($k$) implies the truth of P($k$ + 1).
So by the induction hypothesis

$\boldsymbol{M}^n = \begin{pmatrix} n+1 & -n \\ n & 1-n \end{pmatrix}$, $n \in \mathbb{Z}^+$. ***(R1)***

***[7 marks]***

***Total [16 marks]***

**2.** $$z = \frac{\left(\cos\frac{\pi}{4} - \mathrm{i}\sin\frac{\pi}{4}\right)^2 \left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)^3}{\left(\cos\frac{\pi}{24} - \mathrm{i}\sin\frac{\pi}{24}\right)^4}$$

(a) (i) $|z| = 1$ ***(A1)***

(ii) $$z = \frac{\left(\cos\left(\frac{-\pi}{4}\right) + \mathrm{i}\sin\left(\frac{-\pi}{4}\right)\right)^2 \left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)^3}{\left(\cos\left(\frac{-\pi}{24}\right) + \mathrm{i}\sin\left(\frac{-\pi}{24}\right)\right)^4}$$ ***(M1)***

$$\arg z = 2\left(\frac{-\pi}{4}\right) + 3\left(\frac{\pi}{3}\right) - 4\left(\frac{-\pi}{24}\right)$$ ***(M1)***

$$= \frac{-\pi}{2} + \pi + \frac{\pi}{6}$$

$$= \frac{2\pi}{3}$$ ***(A1)***

**OR**

$\arg z = \frac{2\pi}{3}$ or 2.09 radians ***(G3)***

***[4 marks]***

(b) $$z^3 = \left[1\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)\right]^3$$ ***(M1)***

$= \cos 2\pi + \mathrm{i}\sin 2\pi$ ***(M1)***

$= 1 + 0\mathrm{i}$

$= 1$ ***(AG)***

***[2 marks]***

*continued…*

*Question 2 continued*

(c) $(1+2z)(2+z^2) = 2+z^2+4z+2z^3$

$= 2+z^2+4z+2$ (since $z^3 = 1$) ***(M1)***

$= 4+z^2+4z$

**OR**

$= 4+z^2+4z$

$4+\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)+4\cos\left(\frac{2\pi}{3}\right)+4i\sin\left(\frac{2\pi}{3}\right)$ ***(M1)(A1)***

$\frac{3}{2}+\frac{3\sqrt{3i}}{2}$ ***(A1)(A1)***

$= 1+z+z^2+3+3z$

$= 3+3z$ (since $1+z+z^2 = 0$) ***(M1)***

$= 3+3\left(\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}\right)$ ***(M1)***

$= 3+3\left(-\frac{1}{2}+\frac{\mathrm{i}\sqrt{3}}{2}\right)$ ***(A1)***

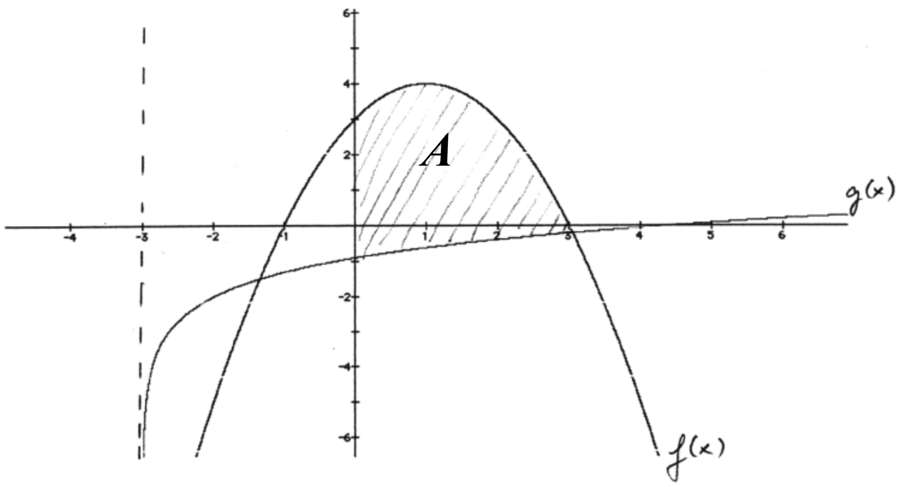
$= \frac{3}{2}+\frac{3\sqrt{3}}{2}\mathrm{i}$ ***(A1)***

***[5 marks]***

***Total [11 marks]***

**3.** (a) ***(A1)(A1)***

[(d) (i) ***(A1)***]




| | |
|---|---|
| **Note:** | Award ***(A1)*** for showing the basic shape of $f(x)$. |
| | Award ***(A1)*** for showing both the vertical asymptote **and** the basic shape of $g(x)$. |

***[2 marks]***

(b) (i) $x = -3$ is the vertical asymptote. ***(A1)***

(ii) $x$-intercept: $x = 4.39$ $(= e^2 - 3)$ ***(G1)***
$y$-intercept: $y = -0.901$ $(= \ln 3 - 2)$ ***(G1)***

***[3 marks]***

(c) $f(x) = g(x)$
$x = -1.34$ or $x = 3.05$ ***(G1)(G1)***

***[2 marks]***

(d) (i) See graph

(ii) Area of $A = \int_0^{3.05} \left(4 - (1-x)^2\right) - \left(\ln(x+3) - 2\right) \mathrm{d}x$ ***(M1)(A1)***

(iii) Area of $A$ = 10.6 ***(G1)***

***[4 marks]***

*continued…*

*Question 3 continued*

(e) $y = f(x) - g(x)$

$y = 5 + 2x - x^2 - \ln(x+3)$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - 2x - \frac{1}{x+3}$ ***(M1)***

Maximum occurs when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$2 - 2x = \frac{1}{x+3}$

$5 - 4x - 2x^2 = 0$

$x = 0.871$ ***(A1)***

$y = 4.63$ ***(A1)***

**OR**

Vertical distance is the difference $f(x) - g(x)$. ***(M1)***

Maximum of $f(x) - g(x)$ occurs at $x = 0.871$. ***(G1)***

The maximum value is 4.63. ***(G1)***

***[3 marks]***

***Total [14 marks]***

**4.** (i) (a) $\vec{AB} = \begin{pmatrix} 5 \\ -10 \\ 25 \end{pmatrix}$. Direction vector of line is $1:-2:5$.
(Accept any multiple of $1:-2:5$) ***(M1)***

Therefore the equation of $l$ in parametric form is
$x = \lambda + 1,\ y = -2\lambda + 3,\ z = 5\lambda - 17$ ***(A1)(A1)(A1)***
(or $x = \lambda + 6,\ y = -2\lambda - 7,\ z = 5\lambda + 8$, or any equivalent **parametric** form)

***[4 marks]***

(b) P on $\ell \Rightarrow$ P can be written as $(p+1, -2p+3, 5p-17)$.

$\vec{OP} \perp \ell \Rightarrow \begin{pmatrix} p+1 \\ -2p+3 \\ 5p-17 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = 0$ ***(M1)***

$p + 1 + 4p - 6 + 25p - 85 = 0$ ***(A1)***

$30p = 90 \ \Rightarrow p = 3$

Therefore P is $(4, -3, -2)$ ***(A1)***

***[3 marks]***

(ii) (a) $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ ***(M1)***

Consider $\dfrac{dy}{dx} = \dfrac{(3y^2 + x^2) \div x^2}{2xy \div x^2}$ ***(M1)***

$= \dfrac{\frac{3y^2}{x^2} + 1}{\frac{2y}{x}} \quad \left( y = vx \Rightarrow v = \dfrac{y}{x} \right)$

$= \dfrac{3v^2 + 1}{2v}$ ***(A1)***

Therefore $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx} = \dfrac{3v^2 + 1}{2v}$ ***(AG)***

***[3 marks]***

(b) $x\dfrac{dv}{dx} = \dfrac{3v^2 + 1}{2v} - v,\ x\dfrac{dv}{dx} = \dfrac{v^2 + 1}{2v}$

$\displaystyle\int \frac{dx}{x} = \int \frac{2v}{v^2 + 1} dv$ ***(M1)***

$\ln x + \ln C = \ln(v^2 + 1)$ ***(A1)***

$y = 2, x = 1 \Rightarrow \ln C = \ln 5$ ***(A1)***

$\ln x + \ln 5 = \ln\left( \dfrac{y^2}{x^2} + 1 \right)$ ***(A1)***

(or $5x = \dfrac{y^2}{x^2} + 1$ or $5x^3 = y^2 + x^2$ or $y = x\sqrt{5x - 1}$ or any equivalent form.)

| |
|---|
| **Note:** Do not penalize if the answer is given as $y = \pm x\sqrt{5x - 1}$. |

***[4 marks]***

***Total [14 marks]***

**5.** (a) (i) To be independent $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$ ***(R1)***

$\mathrm{P}(A) \times \mathrm{P}(B) = (0.85)(0.60)$

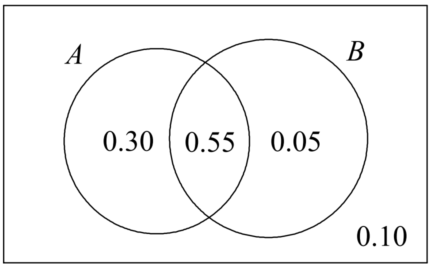
$= 0.51$ ***(A1)***

but $\mathrm{P}(A \cap B) = 0.55$

$\mathrm{P}(A \cap B) \neq \mathrm{P}(A) \times \mathrm{P}(B)$

Hence $A$ and $B$ are not independent. ***(AG)***

(ii)




$\mathrm{P}(B' | A) = \dfrac{\mathrm{P}(B' \cap A)}{\mathrm{P}(A)}$ ***(M1)***

$= \dfrac{0.30}{0.85}$ ***(M1)***

$= \dfrac{6}{17}$ $(= 0.353)$ ***(A1)***

***[5 marks]***

(b) Probability of 2 electricians and 1 plumber $= \dfrac{\binom{3}{2}\binom{2}{1}\binom{5}{2}}{\binom{10}{5}}$ ***(M1)***

$= \dfrac{60}{252}\left(= \dfrac{5}{21} = 0.238\right)$ ***(A1)***

**OR**

Probability of 2 electricians and 1 plumber $= \dfrac{5!}{2!2!}\left(\dfrac{3}{10}\right)\left(\dfrac{2}{9}\right)\left(\dfrac{2}{8}\right)\left(\dfrac{5}{7}\right)\left(\dfrac{4}{6}\right)$ ***(M1)***

$= \dfrac{5}{21}$ $(= 0.238)$ ***(A1)***

***[2 marks]***

*continued…*

*Question 5 continued*

(c) $X$ = number of hours worked.

$X \sim \mathrm{N}(42, \sigma^2)$

$\mathrm{P}(X \geq 48) = 0.10$ ***(AG)***

$\mathrm{P}(X < 48) = 0.90$ ***(M1)***

$\Phi(z) = 0.90$

$z = 1.28$ ***(A1)***

$(z = 1.28155)$

(Answers given to more than 3 significant figures will be accepted.)

$z = \dfrac{X - \mu}{\sigma} \Rightarrow 1.28 = \dfrac{48 - 42}{\sigma}$ ***(M1)***

$\Rightarrow \sigma = 4.69$ (Accept $\sigma = 4.68$) ***(A1)***

$\mathrm{P}(X > 40) = \mathrm{P}\left(Z > \dfrac{40 - 42}{4.69}\right)$ ***(M1)***

$= 0.665$ ***(A1)***

**OR**

$\mathrm{P}(X > 40) = 0.665$ ***(G2)***

Therefore, the probability that one plumber works more than 40 hours per week is 0.665.

The probability that both plumbers work more than 40 hours per week

$= (0.665)^2$ ***(M1)***

= 0.443 (Accept 0.442 or 0.444) ***(A1)***

***[8 marks]***

***Total [15 marks]***

**6.**

**Note:** Accept answers to an accuracy of at least 4 significant figures – do **not** apply AP.

(i) $\mathrm{P}(X=0)=0.04979$, $\mathrm{P}(X=1)=0.14936$, $\mathrm{P}(X=2)=0.22404$,
$\mathrm{P}(X=3)=0.22404$, $\mathrm{P}(X=4)=0.16803$ ***(M1)***
Sum $=0.81526$ ***(A1)***

**OR**

$\mathrm{P}(X \le 4)=0.8153$ (accept 0.8152) ***(G2)***

Hence, $\mathrm{P}(X \ge 5)=1-0.8153=0.1847$ (accept 0.1848) ***(M1)(A1)***

**OR**

$\mathrm{P}(X \ge 5)=1-\mathrm{P}(X \le 4)=0.1847$ ***(G4)***

***[4 marks]***

(ii) The variance is unknown so the *t*-distribution must be used. ***(C1)***
$\bar{x}=2.705$ and $s_n=0.003657$ ***(A1)(A1)***
Hence the confidence interval is given by:

$$\bar{x}-t_{0.975;15}\frac{s_n}{\sqrt{15}},\ \bar{x}+t_{0.975;15}\frac{s_n}{\sqrt{15}}$$ ***(M1)***

**OR**

using $s_{n-1}=0.003777$

$$\bar{x}-t_{0.975;15}\frac{s_{n-1}}{\sqrt{16}},\ \bar{x}+t_{0.975;15}\frac{s_{n-1}}{\sqrt{16}}$$

Since $t_{0.975;15}=2.131$ this gives $[2.703, 2.707]$ ***(A1)(A1)***

**Note:** Award ***(G3)*** for the correct interval, without explanation.

***[6 marks]***

(iii) (a) The data can be described by the following table

| score | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| observed | 27 | 12 | 16 | 25 | 26 | 14 |
| expected | **20** | **20** | **20** | **20** | **20** | **20** |

***(A1)***

$\mathrm{H}_0$ : the die is fair.
$\mathrm{H}_1$ : the die is not fair. ***(C1)***

$$\chi^2_{\text{calc}}=\frac{(20-27)^2}{20}+\frac{(12-20)^2}{20}+\frac{(16-20)^2}{20}+\frac{(25-20)^2}{20}+\frac{(26-20)^2}{20}+\frac{(14-20)^2}{20}$$
$$=11.3$$ ***(A1)***

(i) $\chi^2_{0.95;5}=11.07$ ***(A1)***
Hence, since 11.3 > 11.07 at the 5 % level we must accept $\mathrm{H}_1$. ***(R1)***

(ii) $\chi^2_{0.99;5}=15.086$ ***(A1)***
Hence, since 11.3 < 15.086, at the 1 % level, there is not enough evidence to conclude that the die is not fair (and hence we accept $\mathrm{H}_0$). ***(R1)***

***[7 marks]***

*continued...*

*Question 6(iii) continued*

(b) Let $\alpha$ denote the significance level. If $\chi^2_{\text{calc}}$ is greater than $\chi^2_{\alpha,n-1}$ then it means that the probability of obtaining the results obtained is less than $\alpha$ if $H_0$ is correct. ***(R3)***

| **Note:** Award ***(R3)*** for any correct explanation. Use discretion to award ***(R2)*** or ***(R1)***. |
|---|

***[3 marks]***

(iv) The data can be described by the following table

| | Doctors | Engineers | Lawyers | Businessmen | total |
|---|---|---|---|---|---|
| took up | 63 | 42 | 35 | 68 | 208 |
| did not take up | **73** | **76** | **61** | **82** | 292 |
| total | 136 | 118 | 96 | 150 | 500 |

***(A3)***

| **Note:** Award ***(A3)*** for all correct **bold** entries, ***(A2)*** for 3 correct, ***(A1)*** for 2 correct, ***(A0)*** otherwise. |
|---|

$H_0$ : the percentage is the same in all 4 professions (or equivalent). ***(C1)***
$H_1$ : the percentage is not the same in all 4 professions. ***(C1)***

| **Note:** Do **not** accept $H_1$ : all professions have a different percentage. |
|---|

**METHOD 1**

$p = 0.179$ ***(G3)***
$0.179 > 0.05$ so accept $H_0$ ***(A1)(R1)***

**METHOD 2**

$$\chi^2_{\text{calc}} = \frac{(63-56.6)^2}{56.6} + \frac{(42-49.1)^2}{49} + \frac{(35-39.9)^2}{39.9} + \frac{(68-62.4)^2}{62.4} + \frac{(73-79.4)^2}{79.4}$$
$$+ \frac{(76-68.9)^2}{68.9} + \frac{(61-56.1)^2}{56.1} + \frac{(82-87.6)^2}{87.6}$$ ***(M1)***

$= 4.888$ (sensitive to time of rounding) ***(A1)***

**OR**

$\chi^2_{\text{calc}} = 4.907$ ***(G2)***

$\chi^2_{0.95;\,3} = 7.815$ ***(A1)***
Since $4.888 < 7.815$, $H_0$ is accepted ***(A1)(R1)***

***[10 marks]***

***Total [30 marks]***

**7.** (i) (a)

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

***(A1)***

***[1 mark]***

(b)

| $*$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $b$ | $a$ | $d$ | $c$ |
| $b$ | $a$ | $b$ | $c$ | $d$ |
| $c$ | $d$ | $c$ | $a$ | $b$ |
| $d$ | $c$ | $d$ | $b$ | $a$ |

**OR**

| $*$ | $b$ | $d$ | $a$ | $c$ |
|---|---|---|---|---|
| $b$ | $b$ | $d$ | $a$ | $c$ |
| $d$ | $d$ | $a$ | $c$ | $b$ |
| $a$ | $a$ | $c$ | $b$ | $d$ |
| $c$ | $c$ | $b$ | $d$ | $a$ |

***(A4)***

**Notes:** There are many other correct solutions, with a different ordering of the rows and columns possible.
Award ***(A4)*** if all entries are correct, ***(A3)*** if all but 1 entry are correct,
***(A2)*** if all but 2 entries are correct, ***(A1)*** if all but 3 entries are correct.

***[4 marks]***

(ii) (a) $aRa$ since $a^2 - a^2 = 0 \equiv 0 (\text{mod} 5)$ ***(A1)***

$aRb \Rightarrow bRa$ since $a^2 - b^2 = 0 (\text{mod} 5) \Rightarrow b^2 - a^2 \equiv 0 (\text{mod} 5)$ ***(A1)***

$aRb$ and $bRc \Rightarrow aRc$ since $a^2 - b^2 \equiv 0 (\text{mod} 5)$ and $b^2 - c^2 \equiv 0 (\text{mod} 5)$

$\Rightarrow a^2 - c^2 = a^2 - b^2 + b^2 - c^2 \equiv 0 (\text{mod} 5)$ ***(A2)***

Hence $R$ is an equivalence relation. ***(AG)***

***[4 marks]***

(b) (i) It is the set of all the elements $b$ of $Y$ such that $bRa$. (or equivalent) ***(C2)***

(ii) $\{5, 10\}$ ***(A1)***

$\{1, 4, 6, 9\}$ ***(A1)***

$\{2, 3, 7, 8\}$ ***(A1)***

***[5 marks]***

(iii) There is $\binom{n}{0}$ empty subset. ***(A1)***

There are $\binom{n}{1}$ subsets with 1 element. ***(A1)***

There are $\binom{n}{2}$ subsets with 2 elements.

. . . . . . .

There are $\binom{n}{k}$ subsets with $k$ elements. ***(A1)***

So in total there are $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}$ ***(M1)(A1)***

$= (1+1)^n = 2^n$ subsets. ***(A1)(AG)***

**OR**

Since each of the $n$ elements in set $X$ can be either included in the subset or not, there are $2^n$ possible subsets. ***(A6)***

***[6 marks]***

*continued…*

*Question 7 continued*

(iv) (a) $\begin{pmatrix} a & b & c & d \\ b & d & a & c \end{pmatrix}$ ***(A1)***

***[1 mark]***

(b) $\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix};\begin{pmatrix} a & b & c & d \\ b & a & c & d \end{pmatrix}$ ***(A2)***

| **Note:** There are many correct answers for the second permutation. |
|---|

***[2 marks]***

(c) $\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}$

$\begin{pmatrix} a & b & c & d \\ b & c & d & a \end{pmatrix};\begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix};\begin{pmatrix} a & b & c & d \\ d & a & b & c \end{pmatrix}$ ***(A1)(A1)(A1)***

Let $p$, $q$, $r$, $s$ be the four permutations in the subgroup. Closure is shown by the group table, *i.e.* ***(M1)***

| | $p$ | $q$ | $r$ | $s$ |
|---|---|---|---|---|
| $p$ | $p$ | $q$ | $r$ | $s$ |
| $q$ | $q$ | $r$ | $s$ | $p$ |
| $r$ | $r$ | $s$ | $p$ | $q$ |
| $s$ | $s$ | $p$ | $q$ | $r$ |

***(A1)***

Inverse: each element has an inverse, ***(M1)***
*i.e.* $p^{-1} = p, q^{-1} = s, r^{-1} = r, s^{-1} = q$ . ***(A1)***

| **Note:** There are other possible answers. |
|---|

***[7 marks]***

***Total [30 marks]***

**8.** (i) (a) A finite sequence is defined as follows: $y_0 = a$; $y_1 = b$;
for $n \geq 0$, $y_{n+2}$ is the remainder of the division of $y_n$ by $y_{n+1}$. ***(R1)***
After a finite number of steps, the sequence reaches 0. ***(R1)***
Then if *m* is the smallest integer such that $y_{m+1} = 0$, $y_m$ is the gcd of *a* and *b*. ***(R1)***

| **Note:** Award ***(R3)*** for correct alternative explanations. |
|---|

***[3 marks]***

(b) $364 = 2 \times 154 + 56$
$154 = 2 \times 56 + 42$
$56 = 1 \times 42 + 14$
$42 = 3 \times 14 + 0$ ***(A1)***
Hence $d = 14$ ***(A1)***

$14 = 56 - 42$
$= 56 - (154 - 2 \times 56) = 3 \times 56 - 154$ ***(M1)***
$= 3 \times (364 - 2 \times 154) - 154 = 3 \times 364 - 7 \times 154$
Hence $x = 3$ and $y = -7$. ***(A1)(A1)***

***[5 marks]***

(ii) The characteristic equation is $\lambda^2 - 2\lambda - 2 = 0$ and its roots are $1 + \sqrt{3}$ and $1 - \sqrt{3}$. ***(A1)***
Hence $y_n = C_1(1+\sqrt{3})^n + C_2(1-\sqrt{3})^n$ ***(A1)***
where $C_1$ and $C_2$ are constants to be determined by the initial conditions:
$C_1 + C_2 + \sqrt{3}(C_1 - C_2) = 1$
$4(C_1 + C_2) + 2\sqrt{3}(C_1 - C_2) = 3$ ***(A1)***
Hence $C_1 = \dfrac{\sqrt{3}(1+\sqrt{3})}{12}$; $C_2 = -\dfrac{\sqrt{3}(1-\sqrt{3})}{12}$ ***(A1)***
so that $y_n = \dfrac{\sqrt{3}}{12}\left((1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1}\right)$ $\left(\text{or } y_n = 0.144\left(2.73^{n+1} - (-0.732)^{n+1}\right)\right)$ ***(A1)***

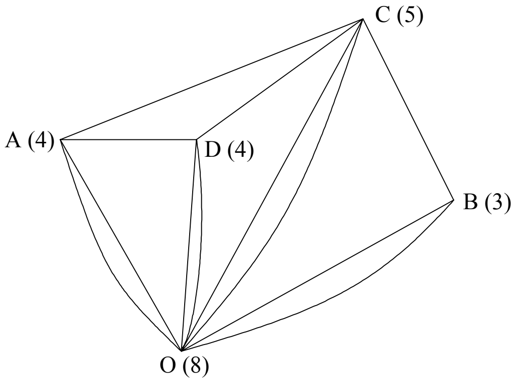
| **Note:** Award final ***(A1)*** for an answer left in the following form $$y_n = \left(\frac{3+\sqrt{3}}{12}\right)\left(1+\sqrt{3}\right)^n + \left(\frac{3-\sqrt{3}}{12}\right)\left(1+\sqrt{3}\right)^n$$ |
|---|

***[5 marks]***

*continued...*

*Question 8 continued*

(iii) (a)




***(A2)***

***[2 marks]***

(b) Yes. The graph has exactly two vertices (B and C) with odd degree. ***(A1)(R1)***
It means that there is a path (starting at B or C) that will go once and only once through every door. ***(C2)***

***[4 marks]***

(c) Yes. $O \to D \to A \to C \to B \to O$ is a Hamiltonian cycle. ***(A1)(R1)***
It means that there is a path (starting anywhere) that will go once and only once through every room before returning to its starting point. ***(C2)***

***[4 marks]***

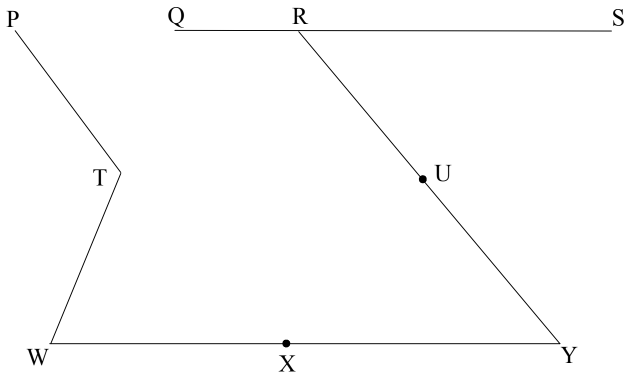
(iv) To every vertex (except the first) there corresponds one and only one edge. ***(R2)***

***[2 marks]***

(v) (a) Starting from a given vertex, **choose some edge incident** to that vertex and **repeat the procedure from the new vertex** reached by the chosen edge until **no more new edges are available and then backtrack** until an untried edge is available and proceed as before. ***(C1)(C1)(C1)***

***[3 marks]***

(b) $P \to T \to W \to X \to Y \to U \to R \begin{smallmatrix} \nearrow S \\ \searrow Q \end{smallmatrix}$ ***(A1)***




***(A1)***

**Note:** There are other correct answers.

***[2 marks]***

***Total [30 marks]***

**9.** (i) (a) $\tan\left(\frac{\pi}{4}+k\pi\right)=\tan\frac{\pi}{4}=1$ since tan has period π. ***(A1)***

***[1 mark]***

(b) (i) In the interval $\left[\frac{\pi}{4},\frac{\pi}{2}\right[$ (which contains 1) $f \geq 0$ and $f'' > 0$ so that the sequence decreases to the only solution of the equation in that interval, namely $\frac{\pi}{4}$. ***(R2)***

| **Note:** Award ***(R2)*** if convergence is shown by a convincing sketch. |
|---|

(ii) $x_1 = 1;\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ ***(C1)***

In this case we get: $x_1 = 1;\ x_{n+1} = x_n - \cos x_n(\sin x_n - \cos x_n)$ ***(C1)***

Hence: $x_1 = 1$

$x_2 = 0.8372778683$

$x_3 = 0.7881802928$

$x_4 = 0.785405918$

$x_5 = 0.7853981635$

$x_6 = 0.7853981634$ ***(A2)***

Therefore $\pi \approx 4x_6 \approx 3.14159$. ***(M1)(A1)***

***[8 marks]***

(ii) Let $f(x) = 2+\cos x$ on the interval $\left[0,\frac{\pi}{2}\right]$.

(a) $\left|f^{(4)}(x)\right| = \left|\cos x\right| \leq 1$ everywhere. ***(A1)***

***[1 mark]***

(b) $A = \int_0^{\frac{\pi}{2}}(2+\cos x)\,\mathrm{d}x$ ***(M1)(A1)***

$= 1+\pi$ ***(AG)***

| **Note:** Evidence of integration required for ***(A1)*** to be awarded. |
|---|

***[2 marks]***

*continued…*

*Question 9(ii) continued*

(c) (i) $y_0 = 3,\ y_{10} = 2$

$\sum_1^5 y_{2k-1} = 13.19622(661)$ ***(A1)***

$\sum_1^4 y_{2k} = 10.65687(5764)$ ***(A1)***

Hence from Simpson's rule we get:

$A \approx (5 + 2\times 10.656875764 + 4\times 13.19622661)\dfrac{\pi}{60}$ ***(M1)***

$= 1.318310966\pi$

$\Rightarrow k = 1.318311$ ***(A1)***

**OR**

$A = 1.318310966\pi$ ***(G3)***

$\Rightarrow k = 1.318311$ ***(A1)***

**OR**

$k = 1.318311$ ***(G4)***

(ii) $\left|1+\pi - k\pi\right| = \left|1+\pi - 1.318311\pi\right| \le \left(\dfrac{4}{20}\right)^4 \left(\dfrac{4}{2}\right)\left(\dfrac{1}{180}\right)$ ***(M1)(AG)***

(iii) $\left|1+\pi - k\pi\right| \le 18\times 10^{-6}$ ***(A1)***

$\Rightarrow 1 - 18\times 10^{-6} \le 0.318310966\pi \le 1 + 18\times 10^{-6}$

that is $3.141526146 < \pi < 3.141637847$ ***(A1)(A1)***

**OR**

Accuracy in the bounds may vary in candidates' solutions, for example

$\left|1+\pi - k\pi\right| \le 1.78\times 10^{-5}$ ***(A1)***

$1 - 1.78\times 10^{-5} \le 0.318311\pi \le 1 + 1.78\times 10^{-5}$

$3.14153 \le \pi \le 3.14164$ ***(A1)(A1)***

This result leads to the correct approximation needed in part (iv)

(iv) $\pi = 3.142$ ***(A3)***

***[11 marks]***

(iii) (a) $S_{2n} = \sum_{k=1}^{2n} \dfrac{1}{k} = S_n + \sum_{k=n+1}^{2n} \dfrac{1}{k} \ge S_n + \sum_{k=n+1}^{2n} \dfrac{1}{2k} = S_n + \dfrac{1}{2}$ ***(M1)(A1)(AG)***

***[2 marks]***

*Question 9(iii) continued*

(b) The sequence $S_n$ is increasing and it follows from (a) that

$S_{2^n} \geq S_{2^{(n-1)}} + \frac{1}{2}$ ***(A1)***

similarly $S_{2^{(n-1)}} \geq S_{2^{(n-2)}} + \frac{1}{2}, S_{2^{(n-2)}} \geq S_{2^{(n-3)}} + \frac{1}{2}$

$\Rightarrow S_{2^n} \geq S_{2^{(n-1)}} + \frac{1}{2} \geq S_{2^{(n-2)}} + \frac{1}{2} + \frac{1}{2} \geq \ldots$ ***(M1)***

*i.e.* $S_{2^n} \geq S_{2^{(n-1)}} + \frac{1}{2} \geq S_{2^{(n-2)}} + \frac{2}{2} \geq \ldots \geq S_1 + \frac{n}{2}$

$S_1 = 1 \Rightarrow S_1 + \frac{n}{2} = \frac{2+n}{2}$

$\Rightarrow S_{2^n} \geq \frac{2+n}{2}$, ***(A1)***

so that there are elements of the sequence that are arbitrarily large and therefore the sequence cannot converge. ***(C1)(R1)***

***[5 marks]***

***Total [30 marks]***

**10.** (i) (a) The conic section is a parabola. ***(A1)***

It is so because the expression may be written in the form $y = ax^2 + bx + c$ . ***(R2)***

***[3 marks]***

(b) The gradient of the tangent to the parabola at a point $(x, y)$ is given by

$$m = \frac{dy}{dx} = 9 - 8x$$ ***(M1)(A1)***

Hence, since the point is on the parabola $y = 5 + 9x - 4x^2$ and since it is on the line we must have $y = (9 - 8x)x + c$ . Therefore: ***(M1)***

$$c = 4x^2 + 5$$ ***(A1)***

$$= 4\left(\frac{9-m}{8}\right)^2 + 5 \quad (\text{or } m^2 - 18m - 16c + 161 = 0)$$ ***(A1)***

***[5 marks]***

(ii) (a) **Ceva's Theorem:** If three concurrent lines are drawn through the vertices A, B, C of a triangle ABC to meet the opposite sides at D, E, F, respectively, then

$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = +1$$ ***(C2)***

**Converse (Corollary):** If D, E, F are points on [BC], [CA], [AB], respectively such that $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = +1$, then [AD], [BE] and [CF] are concurrent. ***(C2)***

***[4 marks]***

(b) If (CM) is a bisector of $B\hat{C}A$ then $\frac{AM}{MB} = \frac{CA}{BC}$. ***(A1)***

Similarly, if (AN) is a bisector of $C\hat{A}B$ then $\frac{BN}{NC} = \frac{AB}{CA}$. ***(A1)***

Finally, if (BP) is a bisector of $A\hat{B}C$ then $\frac{CP}{PA} = \frac{BC}{AB}$. ***(A1)***

Hence $\frac{AM}{MB} \times \frac{BN}{NC} \times \frac{CP}{PA} = 1$ ***(A1)***

and in view of the converse of Ceva's theorem the three bisectors are concurrent. ***(M1)***

***[5 marks]***

*continued…*

*Question 10 continued*

(iii) (a) Let $(n, 0)$ be the coordinates of the point N.

$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -a\sin\theta, \frac{\mathrm{d}y}{\mathrm{d}\theta} = b\cos\theta$ ***(A1)***

Hence the gradient of the line orthogonal to $\ell$ at the point $(x, y)$ is $\frac{a(\tan\theta)}{b}$

so that the equation of the line is given by $y = \frac{(x-n)(a\tan\theta)}{b}$. ***(M1)(A1)***

Since $(a\cos\theta, b\sin\theta)$ is on that line we get:

$b\sin\theta = \frac{(a\cos\theta - n)(a\tan\theta)}{b}$ ***(A1)***

$\Rightarrow n = \frac{(a^2 - b^2)\cos\theta}{a} \Rightarrow \mathrm{N}\left(\frac{(a^2 - b^2)\cos\theta}{a}, 0\right)$ ***(A1)***

***[5 marks]***

(b) Since $\mathrm{F_1O} = \mathrm{OF_2} = c = \sqrt{a^2 - b^2}$, (properties of ellipse) ***(M1)***

$(\mathrm{PF_1})^2 = b^2\sin^2\theta + (a\cos\theta + c)^2 = (a + c\cos\theta)^2$

so that $\mathrm{PF_1} = |a + c\cos\theta| = a + c\cos\theta$ ***(A1)***

$(\mathrm{PF_2})^2 = b^2\sin^2\theta + (a\cos\theta - c)^2 = (a - c\cos\theta)^2$

so that $\mathrm{PF_2} = |a - c\cos\theta| = a - c\cos\theta$ ***(A1)***

***[3 marks]***

(c) $\mathrm{NF_1} = \left|\left(\frac{c^2}{a}\right)\cos\theta + c\right| = \left(\frac{c}{a}\right)(a + c\cos\theta)$ ***(A1)***

$\mathrm{NF_2} = \left|\left(\frac{c^2}{a}\right)\cos\theta - c\right| = \left(\frac{c}{a}\right)(a - c\cos\theta)$ ***(A1)***

***[2 marks]***

(d) $\frac{\mathrm{NF_1}}{\mathrm{NF_2}} = \frac{a + c\cos\theta}{a - c\cos\theta} = \frac{\mathrm{PF_1}}{\mathrm{PF_2}}$ ***(M1)(A1)***

Hence using the bisector theorem, it follows that (PN) bisects the angle $\mathrm{F_1\hat{P}F_2}$. ***(A1)***

***[3 marks]***

***Total [30 marks]***