MARKSCHEME

May 2002

MATHEMATICS

Higher Level

Paper 2

25 pages

Paper 2 Markscheme

Instructions to Examiners

1 Method of marking

(a) All marking must be done using a **red** pen.

(b) Marks should be noted on candidates' scripts as in the markscheme:

- show the breakdown of individual marks using the abbreviations ***(M1)***, ***(A2)*** *etc*.
- write down each part mark total, indicated on the markscheme (for example, ***[3 marks]***) – it is suggested that this be written at the end of each part, and underlined;
- write down and circle the total for each question at the end of the question.

2 Abbreviations

The markscheme may make use of the following abbreviations:

M Marks awarded for **Method**

A Marks awarded for an **Answer** or for **Accuracy**

G Marks awarded for correct solutions, generally obtained from a **Graphic Display Calculator**, irrespective of working shown

C Marks awarded for **Correct** statements

R Marks awarded for clear **Reasoning**

AG **Answer Given** in the question and consequently marks are **not** awarded

3 Follow Through (ft) Marks

Questions in this paper were constructed to enable a candidate to:

- show, step by step, what he or she knows and is able to do;
- use an answer obtained in one part of a question to obtain answers in the later parts of a question.

Thus errors made at any step of the solution can affect all working that follows. Furthermore, errors made early in the solution can affect more steps or parts of the solution than similar errors made later.

To limit the severity of the penalty for errors made at any step of a solution, **follow through (ft)** marks should be awarded. The procedures for awarding these marks require that all examiners:

(i) penalise an error when it **first occurs**;

(ii) **accept the incorrect answer** as the appropriate value or quantity to be used in all subsequent parts of the question;

(iii) award ***M*** marks for a correct method, and ***A*(ft)** marks if the subsequent working contains no further errors.

Follow through procedures may be applied repeatedly throughout the same problem.

The errors made by a candidate may be: arithmetical errors; errors in algebraic manipulation; errors in geometrical representation; use of an incorrect formula; errors in conceptual understanding.

The following illustrates a use of the **follow through** procedure:

Markscheme		Candidate's Script	Marking	
$ 600 × 1.02	***M1***	Amount earned = $ 600 × 1.02	√	***M1***
= $ 612	***A1***	= $602	×	***A0***
$ (306 × 1.02) + (306 × 1.04)	***M1***	Amount = 301 × 1.02 + 301 × 1.04	√	***M1***
= $ 630.36	***A1***	= $ 620.06	√	***A1*(ft)**

Note that the candidate made an arithmetical error at line 2; the candidate used a correct method at lines 3, 4; the candidate's working at lines 3, 4 is correct.

However, if a question is transformed by an error into a **different, much simpler question** then:

(i) **fewer** marks should be awarded at the discretion of the Examiner;

(ii) marks awarded should be followed by '**(d)**' (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(iii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

4 Using the Markscheme

(a) This markscheme presents a particular way in which each question may be worked and how it should be marked. **Alternative methods** have not always been included. Thus, if an answer is wrong then the working must be carefully analysed in order that marks are awarded for a different method in a manner which is consistent with the markscheme.

In this case:

(i) a mark should be awarded followed by '**(d)**' (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(ii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

Alternative solutions are indicated by **OR**. Where these are accompanied by ***G*** marks, they usually signify that the answer is acceptable from a graphic display calculator without showing working. For example:

Mean = 7906/134 ***(M1)***
= 59 ***(A1)***

OR

Mean = 59 (***G2***)

(b) Unless the question specifies otherwise, accept **equivalent forms**. For example: $\dfrac{\sin\theta}{\cos\theta}$ for $\tan\theta$

These equivalent numerical or algebraic forms may be written in brackets after the required answer.

(c) As this is an international examination, all **alternative forms of notation** should be accepted. For example: 1.7 , 1·7 , 1,7 ; different forms of vector notation such as $\vec{u}$, $\bar{u}$, $\underline{u}$; $\tan^{-1} x$ for arctan x.

5 Accuracy of Answers

There are two types of accuracy errors, incorrect level of accuracy, and rounding errors. Unless the level of accuracy is specified in the question, candidates should be penalized **once only IN THE PAPER** for any accuracy error **(AP)**. This could be an incorrect level of accuracy, or a rounding error. Hence, on the **first** occasion in the paper when a correct answer is given to the wrong degree of accuracy, or rounded incorrectly, maximum marks are **not** awarded, but on **all subsequent occasions** when accuracy errors occur, then maximum marks **are** awarded.

There are also situations (particularly in some of the options) where giving an answer to more than 3 significant figures is acceptable. This will be noted in the markscheme.

(a) **Level of accuracy**

(i) In the case when the accuracy of the answer is **specified in the question** (for example: "find the size of angle A to the nearest degree") the maximum mark is awarded **only if** the correct answer is given to the accuracy required.

(ii) When the accuracy is **not** specified in the question, then the general rule applies:

Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.

(b) **Rounding errors**

Rounding errors should only be penalized at the **final answer** stage. This does **not** apply to intermediate answers, only those asked for as part of a question. Premature rounding which leads to incorrect answers should only be penalized at the answer stage.

Incorrect answers are wrong, and should not be considered under (a) or (b).

Examples

A question leads to the answer 4.6789….

- 4.68 is the correct 3 s.f. answer.
- 4.7, 4.679 are to the wrong level of accuracy, and should be penalised the first time this type of error occurs.
- 4.67 is incorrectly rounded – penalise on the first occurrence.

Note: All these "incorrect" answers may be assumed to come from 4.6789..., even if that value is not seen, but previous correct working is shown. However, 4.60 is wrong, as is 4.5, 4.8, and these should be penalised as being incorrect answers, not as examples of accuracy errors.

Markscheme		Candidate's Script (A)	Marking
(a) $a = 2.31 \times 3.43$ $= 7.9233 = 7.92$ (3 s.f.) (b) $2a = 2 \times 7.9233$ $= 15.8466 = 15.8$ (3 s.f.)	***M1*** ***A1*** ***A1*** ***A1***	(a) $a = 2.31 \times 3.43$ $= 7.9233 = 7.92$ (b) $2a = 2 \times 7.29 = 14.58$ $= 14.5$	***M1*** ***A1*** ***A0*** ***A0*(AP)**
		Total	*2 marks*

Notes: Award ***A1*** for either the exact answer 7.9233 or the 3 s.f. answer 7.92.

In line 3, Candidate A has incorrectly transcribed the answer for part (a), but then performs the calculation correctly, and would normally gain the follow through marks. However, the final answer is incorrectly rounded, and the **AP** applies.

Candidate's Script (B)	Marking		Candidate's Script (C)	Marking
(a) $a = 2.31 \times 3.43 = 7.9233$ $= 7.92$ (b) $2a = 2 \times 7.9233$ $= 15.8466 = 15.85$	***M1*** ***A1*** ***A1*** ***A0*(AP)**		(a) $a = 2.31 \times 3.43 = 7.9233$ $= 7.93$ (b) $2a = 2 \times 7.93$ $= 15.86 \quad = 15.8$	***M1*** ***A0*(AP)** ***A1*(ft)** ***A1*(ft)**
Total	*3 marks*		Total	*3 marks*

Notes: Candidate B has given the answer to part (b) to the wrong level of accuracy, **AP** applies.

Candidate C has incorrectly rounded the answers to both parts (a) and (b), is penalised (**AP**) on the first occurrence (line 2), and awarded follow through marks for part (b).

Candidate's Script (D)	Marking		Candidate's Script (E)	Marking
(a) $a = 2.31 \times 3.43$ $= 7.923 = 7.9$ (b) $2a = 2 \times 7.923$ $= 19.446 = 19.5$	***M1*** ***A0*(AP)** ***A1*(ft)** ***A0***		(a) $a = 2.31 \times 3.43 = 7.923$ $= 7.93$ (b) $2a = 2 \times 7.93$ $= 15.86$	***M1*** ***A0*(AP)** ***A1*(ft)** ***A1*(ft)**
Total	*2 marks*		Total	*3 marks*

Notes: Candidate D has given the answer to part (a) to the wrong level of accuracy, and therefore loses 1 mark (**AP**). The answer to part (b) is wrong.

Candidate E has incorrectly rounded the answer to part (a), therefore loses 1 mark (**AP**), is awarded follow through marks for part (b), and does **not** lose a mark for the wrong level of accuracy.

6 Graphic Display Calculators

Many candidates will be obtaining solutions directly from their calculators, often without showing any working. They have been advised that they must use mathematical notation, not calculator commands when explaining what they are doing. Incorrect answers without working will receive no marks. However, if there is written evidence of using a graphic display calculator correctly, method marks may be awarded. Where possible, examples will be provided to guide examiners in awarding these method marks.

1. (a) (i) $\overrightarrow{AB} = -\boldsymbol{j} + 3\boldsymbol{k}$ and $\overrightarrow{AC} = \boldsymbol{i} + 5\boldsymbol{k}$ ***(A1)(A1)***

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & -1 & 3 \\ 1 & 0 & 5 \end{vmatrix}$$ ***(M1)***

$$= -5\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$$ ***(A1)***

(ii) Area $= \frac{1}{2} \times \left| -5\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k} \right|$ ***(M1)***

$= \frac{\sqrt{35}}{2}$ (accept 2.96) ***(A1)***

[6 marks]

(b) (i) The equation of the plane Π is

$(x, y, z).(-5, 3, 1) = \mathrm{c}$, that is, $-5x + 3y + z = \mathrm{c}$ ***(M1)(A1)***

where $\mathrm{c} = -5 + 9 + 1 = 5$, that is, $-5x + 3y + z = 5$ ***(A1)***

Note: Award ***(M1)(A1)(A0)*** if answer not given in Cartesian form.

(ii) Equations of L are

$$\frac{x-5}{-5} = \frac{y+2}{3} = z - 1$$ ***(M1)(A1)***

Note: Award ***(M1)(A0)*** if answer not given in Cartesian form.

[5 marks]

(c) L meets Π where

$-5(5 - 5\lambda) + 3(3\lambda - 2) + \lambda + 1 = 5$ ***(M1)***

$\Rightarrow \lambda = 1$ ***(A1)***

Point of intersection is (0, 1, 2). (Accept $\boldsymbol{j} + 2\boldsymbol{k}$.) ***(A1)***

[3 marks]

(d) Perpendicular distance is $\sqrt{5^2 + 3^2 + 1^2}$ ***(M1)***

$= \sqrt{35}$ (Accept $\pm\sqrt{35}$ or ± 5.92) ***(A1)***

[2 marks]

Total [16 marks]

2. (a) (i) $2x^2\left(v + x\dfrac{\mathrm{d}v}{\mathrm{d}x}\right) = x^2 + v^2x^2$ ***(M1)***

$v + x\dfrac{\mathrm{d}v}{\mathrm{d}x} = \dfrac{1+v^2}{2}$

$\Rightarrow 2x\dfrac{\mathrm{d}v}{\mathrm{d}x} = (v-1)^2$ ***(A1)(AG)***

(ii) $2\displaystyle\int \frac{\mathrm{d}v}{(v-1)^2} = \int \frac{\mathrm{d}x}{x}$ ***(M1)***

$\dfrac{-2}{v-1} = \ln x + c$ ***(A1)***

$\dfrac{-2}{\frac{y}{x}-1} = \ln x + c$ ***(A1)***

$\dfrac{-2}{\ln x + c} = \dfrac{y}{x} - 1 \Rightarrow \dfrac{-2x}{\ln x + c} = y - x$ ***(A1)***

$\Rightarrow y = x - \dfrac{2x}{\ln x + c}$ ***(AG)***

> **Note:** For other approaches award marks as follows
> ***(M1)*** for separation of variables; ***(A1)*** for correct integration;
> ***(A1)*** for substitution; ***(A1)*** for solving for y.

(iii) $y = 2, x = 1 \rightarrow c = -2$ ***(A1)***

[7 marks]

(b) (i) $x = \mathrm{e}^2$ or $x = 7.39$

> **Note:** Award ***(A0)*** if answer is not given as an equation (just e^2 or 7.39)

(A1)

(ii) $x = \dfrac{2x}{\ln x - 2} \Rightarrow \ln x = 4$ ***(M1)***

$\Rightarrow x = \mathrm{e}^4$ (must be exact) ***(A1)***

> **Note:** Award ***(M0)(G1)*** for $x = 54.6$.

(iii) Area $= 46.7$ ***(G2)***

OR

Area $= \displaystyle\int_1^5 y\,\mathrm{d}x$ ***(M1)***

$= 46.7$ ***(G1)***

[5 marks]

Total [12 marks]

3. (i) (a) Determinant = 0 ***(A1)***

[1 mark]

(b) Using row operations,

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ \lambda - 6 \end{pmatrix}$$ ***(M1)(M1)***

whence $\lambda = 5$. ***(A1)***

Note: There are other methods.

[3 marks]

(c) Put $z = \mu$, so that $y = 1 + \mu$ and $x = 2 - 3\mu$, ***(M1)(A1)(A1)***
or $y = \mu$ so that $z = \mu - 1$ and $x = 5 - 3\mu$.

Note: Award ***(M1)*** for any correct method and ***(A1)*** for each correct equation (ignore the equation that identifies the parameter). Award no marks for particular solutions.

[3 marks]

(ii) (a) Let $n = 1$, $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^1 & 2^1 - 1 \\ 0 & 1 \end{pmatrix}$ ***(A1)***

Thus true for $n = 1$.
Assume true for $n = k$, that is,

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 2^k & 2^k - 1 \\ 0 & 1 \end{pmatrix}$$ ***(M1)***

Consider

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2^k & 2^k - 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 2^{k+1} - 1 \\ 0 & 1 \end{pmatrix}$$ ***(M1)*** ***(A1)***

True for $k \Rightarrow$ true for $k+1$. (Or some correct concluding statement.) ***(A1)***

[5 marks]

(b) $\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad \boldsymbol{A}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$ ***(A1)***

$$n = -1 \Rightarrow \begin{pmatrix} 2^{-1} & 2^{-1} - 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} = \boldsymbol{A}^{-1}$$ ***(R1)***

[2 marks]

Total [14 marks]

4. (a) (i) P(Alan scores 9) $= \frac{1}{9}$ $(= 0.111)$ ***(A1)***

(ii) P(Alan scores 9 and Belle scores 9) $= \left(\frac{1}{9}\right)^2 = \left(\frac{1}{81}\right)$ $(= 0.0123)$ ***(A1)***

[2 marks]

(b) (i) P(Same score) $= \left(\frac{1}{36}\right)^2 + \left(\frac{2}{36}\right)^2 + \ldots + \left(\frac{6}{36}\right)^2 + \ldots + \left(\frac{2}{36}\right)^2 + \left(\frac{1}{36}\right)^2$ ***(M1)***

$= \frac{73}{648}$ $(= 0.113)$ ***(A1)***

(ii) $\mathrm{P}(A > B) = \frac{1}{2}\left(1 - \frac{73}{648}\right)$ ***(M1)***

$= \frac{575}{1296}$ $(= 0.444)$ ***(A1)***

[4 marks]

(c) (i) P(One number $\le x) = \frac{x}{6}$ (with some explanation) ***(R1)***

$\mathrm{P}(X \le x) = \mathrm{P}(\text{All four numbers} \le x) = \left(\frac{x}{6}\right)^4$ ***(M1)(AG)***

(ii) $\mathrm{P}(X = x) = \mathrm{P}(X \le x) - \mathrm{P}(X \le x - 1) = \left(\frac{x}{6}\right)^4 - \left(\frac{x-1}{6}\right)^4$

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{1296}$	$\frac{15}{1296}$	$\frac{\mathbf{65}}{\mathbf{1296}}$	$\frac{\mathbf{175}}{\mathbf{1296}}$	$\frac{\mathbf{369}}{\mathbf{1296}}$	$\frac{671}{1296}$

(A1)(A1)(A1)

Note: Award ***(A3)*** if table is not completed but calculation of E(X) in part (iii) is correct.

(iii) $\mathrm{E}(X) = 1 \times \frac{1}{1296} + 2 \times \frac{15}{1296} + \ldots + 6 \times \frac{671}{1296}$ ***(M1)***

$= \frac{6797}{1296}$ $(= 5.24)$ ***(A1)***

[7 marks]

Total [13 marks]

5. (a) (i) $f'(x)=\dfrac{(2x-1)(x^2+x+1)-(2x+1)(x^2-x+1)}{(x^2+x+1)^2}$ ***(M1)(A1)***

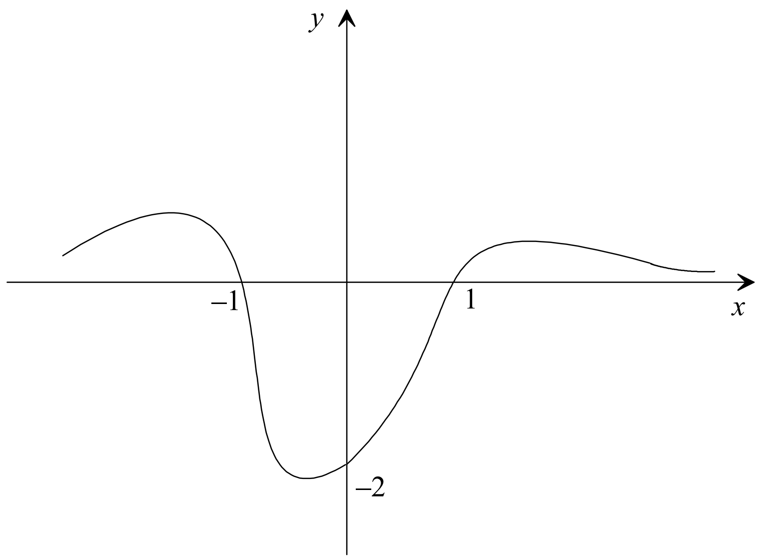
$=\dfrac{2(x^2-1)}{(x^2+x+1)^2}$ ***(A1)***

(ii) $f'(x)=0 \Rightarrow x=\pm 1$

$\mathrm{A}\left(1,\dfrac{1}{3}\right) \quad \mathrm{B}(-1,3) \quad \left(\text{or } \mathrm{A}(-1,3) \quad \mathrm{B}\left(1,\dfrac{1}{3}\right)\right)$ ***(A1)(A1)***

[5 marks]

(b) (i)



(G2)

Note: Award ***(G1)*** for general shape and ***(G1)*** for indication of scale.

(ii) The points of inflexion can be found by locating the max/min on the graph of f'.
This gives $x=-1.53,-0.347,1.88$. ***(G3)***

OR

$f''(x)=\dfrac{-4(x^3-3x-1)}{(x^2+x+1)^3}$ ***(M1)***

$f''(x)=0 \Rightarrow x^3-3x-1=0$ ***(A1)***

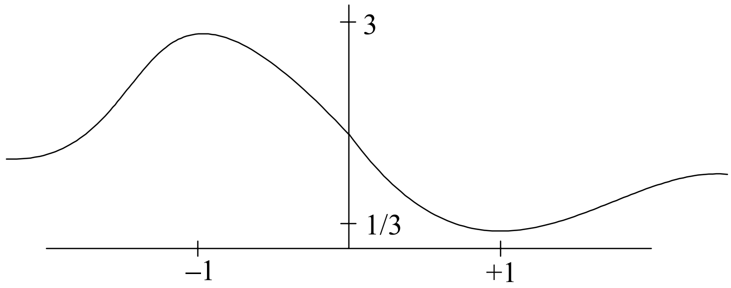
$\Rightarrow x=1.53,-0.347,1.88$ ***(G1)***

[5 marks]

continued…

Question 5 continued

(c) The graph of $y = f(x)$ helps:



(i) Range of f is $\left[\frac{1}{3}, 3\right]$. ***(A1)(A1)***

(ii) We require the image set of $\left[\frac{1}{3}, 3\right]$.

$$f\left(\frac{1}{3}\right) = \frac{\frac{1}{9} - \frac{1}{3} + 1}{\frac{1}{9} + \frac{1}{3} + 1} = \frac{7}{13},\ f(3) = \frac{9 - 3 + 1}{9 + 3 + 1} = \frac{7}{13}$$ ***(M1)***

Range of g is $\left[\frac{1}{3}, \frac{7}{13}\right]$. ***(A1)(A1)***

Note:	Since the question did not specify **exact** ranges accept open intervals or numerical approximations (*e.g.* [0.333, 0.538]).

[5 marks]

Total [15 marks]

6. (i) (a) The given condition implies that

$$\frac{\mu^3}{6}\mathrm{e}^{-\mu} = \mathrm{e}^{-\mu} + \mu\mathrm{e}^{-\mu}$$ ***(M1)***

$\Rightarrow \mu^3 - 6\mu - 6 = 0$ ***(A1)***

$\Rightarrow \mu \simeq 2.8473$ ***(G1)***

[3 marks]

(b) $\mathrm{P}(2 \le X \le 4) = \mathrm{P}(X = 2) + \mathrm{P}(X = 3) + \mathrm{P}(X = 4)$ ***(M1)***

$\mathrm{P}(X = 2) = \dfrac{\mu^2\mathrm{e}^{-\mu}}{2} = 0.235$, $\mathrm{P}(X = 3) = \dfrac{\mu^3\mathrm{e}^{-\mu}}{6} = 0.223$, $\mathrm{P}(X = 4) = \dfrac{\mu^4\mathrm{e}^{-\mu}}{24} = 0.159$ ***(G1)***

Hence $\mathrm{P}(2 \le X \le 4) = 0.617$ ***(A1)***

OR

$\mathrm{P}(2 \le X \le 4) = \mathrm{P}(X \le 4) - \mathrm{P}(X \le 1)$ ***(M1)***

$= 0.8402 - 0.2231$ ***(G1)***

$= 0.617$ ***(G1)***

[3 marks]

(ii) Let W be the weight of a male nurse, $W \sim \mathrm{N}(72, 7.5^2)$

$\overline{W}$ is normally distributed with mean $\mu = 72$, and variance $\sigma^2 = \dfrac{(7.5)^2}{6} = 9.375$. ***(A1)(A1)***

Hence $p = \mathrm{P}(6\overline{W} > 450) = \mathrm{P}\left(\overline{W} \ge \dfrac{450}{6}\right)$. ***(M1)***

$$p = 1 - \Phi(t) \text{ where } t = \frac{\left(\frac{450}{6} - 72\right)}{\sqrt{9.375}} \simeq 0.980$$ ***(A1)***

so that $p \simeq 0.164$ ***(A1)***

OR

$p = 0.164$ ***(G2)***

[5 marks]

continued…

Question 6 continued

(iii) H_0 both varieties have the same mean yield, H_1 varieties have a different mean yield. ***(C1)***

The mean of the yield of the first sample is 9.4. ***(A1)***

The unbiased estimate for the variance of the yield of the first sample is $(0.563)^2 = 0.317$. ***(A1)***

The mean of the yield of the second sample is 8.6. ***(A1)***

The unbiased estimate for the variance of the yield of the second sample is $(0.613)^2 = 0.376$. ***(A1)***

In this case the *t*-test (with 12 degrees of freedom) must be used. ***(C1)***

Then $t = \dfrac{|9.4 - 8.6|}{\sqrt{\dfrac{7\times(0.563)^2 + 5\times(0.613)^2}{12}}\sqrt{\dfrac{1}{8}+\dfrac{1}{6}}} = 2.534$ ***(M1)(A1)***

$t_{0.025,12} = 2.179$ ***(A1)***

Since 2.534 > 2.179 there is ground to reject the hypothesis H_0 ***(R1)***

OR

H_0 both varieties have the same mean yield, H_1 varieties have a different mean yield. ***(C1)***

For stating that a 2-sample *t*-test must be used ***(C1)***

For correct value of *p*-number (0.0262) ***(G6)***

Since 0.0262 < 0.5 there is ground to reject the hypothesis H_0 ***(R2)***

Note:	If *p*-number is incorrect but the work shows intermediate values for mean and variance award marks according to the first method.

[10 marks]

(iv) Let H_0 be the hypothesis that all coins are fair, ***(C1)***

and let H_1 be the hypothesis that not all coins are fair. ***(C1)***

Let *T* be the number of tails obtained, *T* is binomially distributed. ***(M1)***

T	0	1	2	3	4	5	6
f_o	5	40	86	89	67	29	4
f_e	**5**	**30**	**75**	**100**	**75**	**30**	**5**

(A3)

Notes:	Award ***(A2)*** if one entry on the third row is incorrect. Award ***(A1)*** if two entries on the third row are incorrect. Award ***(A0)*** if three or more entries on the third row are incorrect.

$$\chi^2_{\text{calc}} = \frac{(5-5)^2}{5} + \frac{(40-30)^2}{30} + \frac{(86-75)^2}{75} + \frac{(89-100)^2}{100} + \frac{(67-75)^2}{75} + \frac{(29-30)^2}{30} + \frac{(4-5)^2}{5}$$

$= 7.24$ ***(A1)***

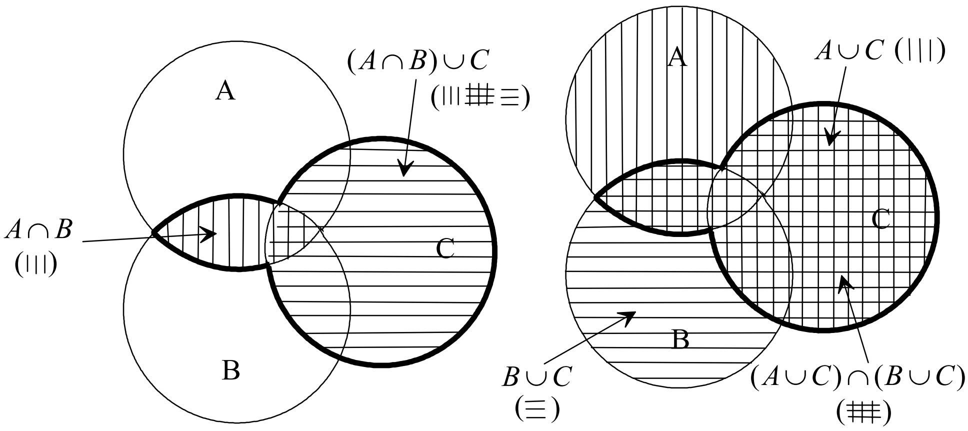
Also $\chi^2_{0.05,6} = 12.592$ ***(A1)***

Since 7.24 < 12.592, H_0 cannot be rejected. ***(R1)***

[9 marks]

Total [30 marks]

7. (i) (a)



That is, $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ ***(M1)(A1)***

[2 marks]

(b) From part (a) $(A' \cap B) \cup C' = (A' \cup C') \cap (B \cup C')$. ***(A1)***

From De Morgan's laws $(A \cap C)' = A' \cup C'$, and $(B' \cap C)' = B \cup C'$ ***(A1)(A1)***

So $(A' \cap B) \cup C' = (A \cap C)' \cap (B' \cap C)'$ ***(AG)***

[3 marks]

(ii) (a) $\forall a \in \mathbb{Z}, a\ R\ a$ ***(A1)***

$\forall a, b \in \mathbb{Z}, a\ R\ b \Rightarrow m$ divides $a - b \Rightarrow m$ divides $b - a \Rightarrow b\ R\ a$ ***(A1)***

$\forall a, b, c \in \mathbb{Z}, a\ R\ b$ and $b\ R\ c \Rightarrow m$ divides $(a - b)$ and m divides $(b - c)$

$\Rightarrow m$ divides $(a - b) + (b - c) \Rightarrow m$ divides $(a - c) \Rightarrow a\ R\ c$ ***(A1)***

Hence R is an equivalence relation. ***(C1)***

[4 marks]

(b) For any reasonable attempt to explain that the equivalence relation partitions the set. ***(C2)***

For either the list of equivalence classes that partition $\mathbb{Z}$ or an attempt to explain that there are m equivalence classes. ***(C2)***

[4 marks]

Question 7 (ii) continued

(c) For an attempt to establish closure. ***(M1)***
Associativity : comes from the associativity of + in $\mathbb{Z}$. ***(A1)***
Commutativity : comes from the commutativity of + in $\mathbb{Z}$. ***(A1)***
There is an identity element : [0]. ***(A1)***
Every element has an inverse $k = 1, \ldots, m-1$ the inverse of [k], is [$m - k$] and [0] is its own inverse ***(A1)***

Notes:	If a reasonable **attempt** on the general case is made to prove any of the above, award the corresponding mark. Award a maximum of ***(A4)*** if the properties above are established for a specific value of m. Award ***(A1)*** if all five properties are listed without any further amplification.

[5 marks]

(d) For indicating knowledge of what a cyclic group is ***(C1)***
For stating $(\mathbb{Z}_m ; +_m)$ ***(A1)***
For identifying [1] as the generator of $(\mathbb{Z}_m ; +_m)$ ***(C1)***
For stating that two cyclic groups of the same order are isomorphic or for attempting to identify an isomorphism. ***(C1)***

[4 marks]

(iii) **If** one of the sets H and K is contained in the other then either $H \cup K = H$ or $H \cup K = K$.
In either case it is a subgroup of $(G, \circ)$. ***(C2)***

Only if:

Conversely, suppose that $(H \cup K, \circ)$ is a subgroup of $(G, \circ)$ and that H is not contained in K. ***(M1)***
Then there exists an element b of H which is not included in K. ***(C1)***
Let a be any element of K.
Then $ab \in H \cup K$ (since $(H \cup K, \circ)$ is a group). ***(C1)***
If $ab \in K$ then $b = a^{-1}ab \in K$ which is a contradiction of our hypothesis. ***(C1)***
Hence $ab \notin K$ and therefore $ab \notin H$ so that $abb^{-1} \in H$ ***(C1)***
which shows that $K \subseteq H$ since a was any element of K. ***(C1)***
Therefore $H \subseteq K$ or $K \subseteq H$. ***(AG)***

OR

Proof by contradiction: ***(M1)***
$K \not\subset H$ then there exists $m \in K, m \notin H$ ***(C1)***
And
$H \not\subset K$ then there exists $n \in H, n \notin K$. ***(C1)***
Suppose $m \circ n \in H$ then $m \circ n \circ n^{-1} \in H$ is a contradiction ***(C1)***
Suppose $m \circ n \in K$ then $n = m^{-1} \circ m \circ n \in K$ is a contradiction ***(C1)***
Hence $m \circ n \notin H \cup K$ a contradiction ***(C1)***
Therefore $H \subseteq K$ or $K \subseteq H$ ***(AG)***

[8 marks]

Total [30 marks]

8. (i) (a) $568 = 2\times 208 + 152;\ 208 = 152 + 56;\ 152 = 2\times 56 + 40;\ 56 = 40 + 16;$ ***(M1)***
$40 = 2\times 16 + 8;\ 16 = 2\times 8$ ***(M1)***
Hence 8 is the greatest common divisor of 568 and 208. ***(A1)***

[3 marks]

(b) $8 = 40 - 2\times 16 = -2\times 56 + 3\times 40 = 3\times 152 - 8\times 56$
$= -8\times 208 + 11\times 152 = 11\times 568 - 30\times 208$ ***(M1)(M1)***
Hence $m = 11$ and $n = 30$. ***(A1)(A1)***

[4 marks]

(ii) Any vertex of K_n has $n-1$ adjacent vertices and therefore its colour must be different from the colour of every other vertex. ***(M1)***
Since there are n vertices, the colouring of K_n requires n colours. ***(R2)***

[3 marks]

(iii) **Each** directed edge leaves one vertex and goes into another so each edge contributes one "ins " and one "out", ***(M1)***
so the total number of "outs" = total number of "ins" = total number of edges ***(C1)***
Hence $S_1 = S_2 = S_3$. ***(AG)***

[2 marks]

(iv) (a) Two graphs *G* and *H* are isomorphic if :
(1) there exists a bijection Φ between the set of all vertices of *H* onto the set of all vertices of *G* ***(C1)***
(2) whenever there are *k* edges between two vertices, say *a* and *b* of *G* there are *k* edges between $\Phi(a)$ and $\Phi(b)$. ***(C2)***

OR

They have the same adjacency matrix. ***(C3)***

[3 marks]

(b) When two graphs are isomorphic, the isomorphism preserves the degree of each vertex. ***(C2)***
In this case they cannot be isomorphic because *G* has a vertex, {B}, of degree 3 while *H* has no vertex of degree 3. ***(A2)***

[4 marks]

(c) $B \to A \to E \to B \to C \to E \to F \to C \to D \to F$ ***(C3)***

Note:	There are alternative correct answers.

[3 marks]

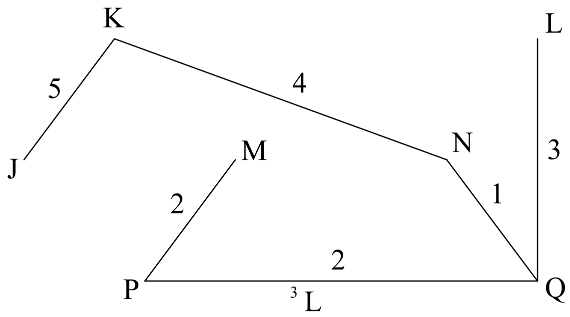
(d) *H* has an Eulerian circuit because all the vertices of *H* have an even degree. ***(C2)***

[2 marks]

continued…

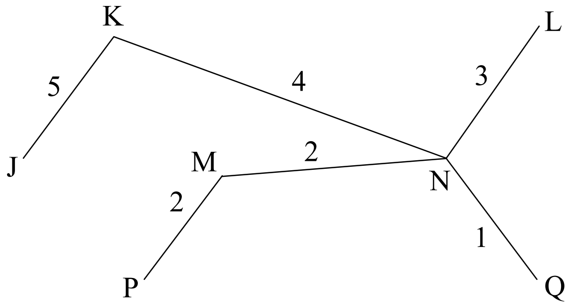
Question 8 continued

(v)



(C4)

OR



(C4)

Total weight = 17 ***(A2)***

Note:	There are other possible spanning trees.

[6 marks]

Total [30 marks]

9. (i) (a) $n\ln\left(1+\frac{1}{n}\right) = n\int_1^{1+\frac{1}{n}} \frac{\mathrm{d}t}{t} = n\frac{1}{c}$ with $1<c<1+\frac{1}{n}$ ***(M1)(A1)***

hence $\frac{1}{c}<1$ so that $n\ln\left(1+\frac{1}{n}\right)<1$ ***(A1)(AG)***

OR

$n\ln\left(1+\frac{1}{n}\right) = n\int_1^{1+\frac{1}{n}} \frac{\mathrm{d}t}{t} = n\frac{1}{c}$ with $1<c<1+\frac{1}{n}$ ***(M1)(A1)***

$\leq n\int_1^{1+\frac{1}{n}} \mathrm{d}t \leq 1$ ***(A1)(AG)***

[3 marks]

(b) Since $\forall s \in]0,4[, s^2-4s+4=(s-2)^2 \geq 0$ ***(A1)***

$\frac{1}{s}+\frac{1}{4-s}=\frac{4}{s(4-s)}\geq 1$ ***(A1)***

Note: Award ***(G1)*** if the inequality is shown by calculator.

[2 marks]

(c) From part (b)

$\int_t^2 \frac{\mathrm{d}s}{s}+\int_t^2 \frac{\mathrm{d}s}{4-s} \geq \int_t^2 \mathrm{d}s$ ***(M1)(A1)***

$\int_t^2 \frac{\mathrm{d}s}{s} = \ln 2 - \ln t$ ***(A1)***

$\int_t^2 \frac{\mathrm{d}s}{4-s} = \ln(4-t)-\ln 2$ ***(A1)***

$\int_t^2 \mathrm{d}s = 2-t$ ***(A1)***

Hence $\ln 2 - \ln t + \ln(4-t) - \ln 2 \geq 2-t$

so that $\ln\left(\frac{4-t}{t}\right) \geq 2-t$ ***(A1)***

[6 marks]

(d) Let $2-t=\frac{2}{2n+1} \Rightarrow 4n+2-2tn-t=2$

$\Rightarrow 4n=t(2n+1)$, where n is a positive integer. ***(M1)***

$\Rightarrow t=\frac{4n}{2n+1} \in]0,2]$. ***(R1)***

Hence replacing this value of t in the inequality obtained in part (c) :

$\ln\left(1+\frac{1}{n}\right) \geq \frac{2}{2n+1}$ ***(A1)(A1)***

Multiplying both sides by n, $\Rightarrow n\ln\left(1+\frac{1}{n}\right) \geq \frac{2n}{2n+1}$ ***(AG)***

[4 marks]

continued...

Question 8 continued

(e) From parts (a) and (d), $\dfrac{2n}{2n+1} \le n\ln\left(1+\dfrac{1}{n}\right) \le 1$ ***(C1)***

and $\Rightarrow \lim\limits_{n\to\infty} \dfrac{2n}{2n+1} = 1$ ***(A1)***

Hence $\Rightarrow \lim\limits_{n\to\infty} n\,1n\left(1+\dfrac{1}{n}\right) = 1$. ***(M1)***

$\lim\limits_{n\to\infty} 1\mathrm{n}\left(1+\dfrac{1}{n}\right)^n = 1$

$\lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^n = \mathrm{e}$ ***(A1)(AG)***

[4 marks]

Note:	Different calculators may give slightly different sequences in parts (ii) and (iii), but the final answer should be the same.

(ii) $x_1 = 2;\ x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_1 = 2;\ x_{n+1} = x_n(2 - 1\mathrm{n}\ x_n)$.

For all $x > 0$ we see that $f'(x) > 0$ and $f''(x) < 0$. Since $f(2) < 0$, the method will yield an increasing sequence which converges to the solution.

In this case we get : $x_1 = 2;\ x_{n+1} = x_n(2 - \log x_n)$.

Hence : $x_1 = 2$

$x_2 = 2.613705639$ ***(G1)***

$x_3 = 2.716243926$ ***(G1)***

$x_4 = 2.718281064$ ***(G1)***

$x_5 = 2.718281828$

$x_6 = 2.718281828$

In $x - 1 = 0 \Rightarrow x = \mathrm{e}$ ***(A1)***

Hence $\mathrm{e} \approx 2.7182818$ ***(A1)***

[5 marks]

continued...

Question 9 continued

(iii) (a) $g(\mathrm{e}) = \mathrm{e} + 2.7 - 2.7 = \mathrm{e}$ ***(A1)***

[1 mark]

(b) For all $x > 1.35$ we see that $|g'(x)| < 1$ so that the method will yield a convergent sequence. ***(M1)(R1)***

$x_1 = 2$

$x_2 = 2.828502612$ ***(G1)***

$x_3 = 2.721184472$ ***(G1)***

$x_4 = 2.718302888$

$x_5 = 2.718281897$

$x_6 = 2.718281829$

$x_7 = 2.718281828$

Hence $\mathrm{e} \approx 2.7182818$ ***(A1)***

OR

$x_5 = 2.718281828$ ***(G4)***

$\Rightarrow \mathrm{e} = 2.7182818$ ***(A1)***

[5 marks]

Total [30 marks]

10. (i) (a) The given equation is equivalent to $\frac{(x-3)^2}{1}+\frac{(y+2)^2}{4}=1$

Hence : centre $=(3,-2)$; ***(A1)***

eccentricity $=\frac{\sqrt{3}}{2}$; ***(A1)***

foci $=\left(3,-2\pm\sqrt{3}\right)$ ***(A1)***

[3 marks]

(b) Since $m=\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4(3-x)}{y+2}$

$\Rightarrow y(y+2)=12x-4x^2$ (1) ***(M1)(A1)***

$4x^2+y^2-24x+4y+36=0$ (2) ***(A1)***

$\Rightarrow 2y-12x+36=0 \Rightarrow y=6(x-3)$

$\Rightarrow 40x^2-216x+288=0$ ***(A1)***

Hence $x=3$ and $x=\frac{12}{5}$; the corresponding values of y are 0 and $-\frac{18}{5}$.

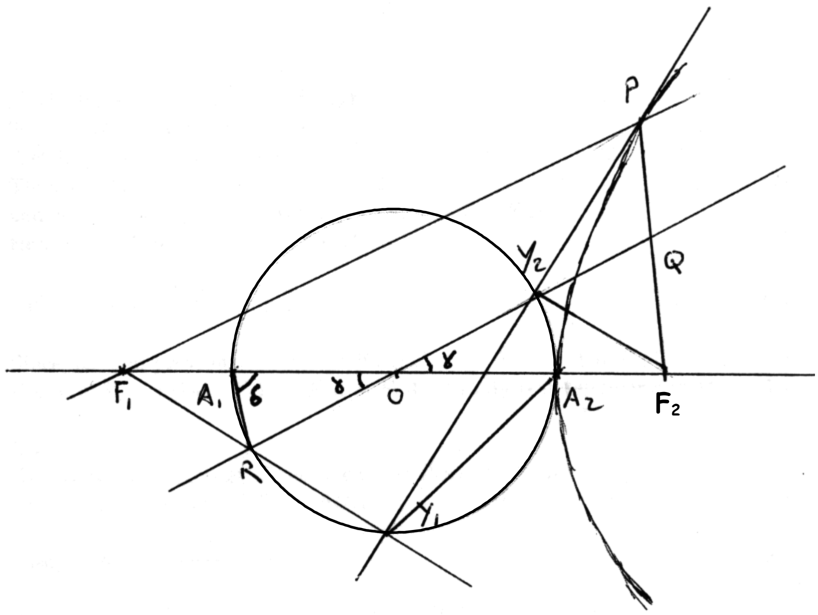
Therefore, the values of m are $0,-\frac{3}{2}$. ***(A2)***

Note: Award ***(G1)*** for approximate values of *m*.

[6 marks]

Question 10 continued

(ii) (a)



Draw through Y_2 a line parallel to (F_1P) that intersects (PF_2) at Q, (F_1F_2) at O and (F_1Y_1) at R.

$(PY)_2$ is the bisector of F_1PF_2 (property of the tangent to a conic) ***(A1)***

$Q\hat{Y}_2P = Y_2\hat{P}F_1 = Y_2\hat{P}Q$ so that ΔQY_2P is isosceles ***(C1)***

and hence $Y_2Q = PQ$

$Q\hat{F}_2Y_2 = \dfrac{\pi}{2} - Q\hat{P}Y_2 = Q\hat{Y}_2F_2$ so that ΔQY_2F_2 is isosceles ***(C1)***

and hence $Y_2Q = QF_2$ ***(A1)***

Therefore $PQ = QF_2$ so that Q is the midpoint of $[PF_2]$. ***(A1)***

ΔF_1PF_2 is similar to ΔOQF_2 so that O is the midpoint of $[F_1F_2]$ and hence of $[A_1A_2]$. ***(A1)***

Consequently ΔF_1RO is congruent to ΔF_2Y_2O so that $OR = OY_2$. ***(A1)***

Hence ΔOY_1Y_2 is isosceles so that $OY_1 = OY_2$. ***(A1)***

Finally $OY_2 = OQ - Y_2Q = \dfrac{PF_1}{2} - \dfrac{PF_2}{2} = \dfrac{A_1A_2}{2}$ ***(A1)***

so that $OY_1 = OY_2 = OA_1 = OA_2$. ***(A1)***

Therefore Y_1 and Y_2 lie on the circle which has $[A_1A_2]$ as diameter. ***(AG)***

[10 marks]

Question 10 (ii) continued

(b) Draw the segments $[A_1R]$ and $[A_2Y_1]$.

Set $\gamma = R\hat{O}A_1$ and $\delta = O\hat{A}_1R$

ΔROA_1 is isosceles (since $OR = OA_1$) so that $\delta = \dfrac{(\pi - \gamma)}{2}$ ***(C1)***

Then $F_1\hat{A}_1R = \pi - \delta = \pi - \dfrac{(\pi - \gamma)}{2} = \dfrac{(\pi + \gamma)}{2}$ ***(C1)***

and $A_2\hat{Y}_1F_1 = A_2\hat{Y}_1A_1 + A_1\hat{Y}_1F_1 = \dfrac{\pi}{2} + \dfrac{\gamma}{2} = F_1\hat{A}_1R$ ***(C1)***

Hence ΔF_1A_1R is similar to $\Delta F_1Y_1A_2$ ***(C1)***

and therefore $\dfrac{F_1R}{F_1A_1} = \dfrac{F_1A_2}{F_1Y_1}$ ***(C1)***

Since, as seen previously, ΔF_1RO is congruent to ΔF_2Y_2O it follows that $F_2Y_2 = F_1R$ ***(C1)***

Hence $F_1Y_1 \times F_2Y_2 = F_1Y_1 \times F_1R = F_1A_1 \times F_1A_2$, which is a constant, independent of the position of P. ***(AG)***

Note:	There are other possible solutions, using analytic geometry.

[6 marks]

continued…

Question 10 continued

(iii) Draw [OC] and [OD].

Then $D\hat{A}B + A\hat{B}D + B\hat{D}A = 5D\hat{A}B = \pi$, so that $D\hat{A}B = \frac{\pi}{5}$ ***(A1)***

Also, $D\hat{O}C = 2D\hat{A}B = \frac{2\pi}{5}$

And $C\hat{D}O = O\hat{C}D$ and $C\hat{D}O + D\hat{O}C + O\hat{C}D = \pi$

Hence $C\hat{D}O = \frac{3\pi}{10}$ ***(A1)***

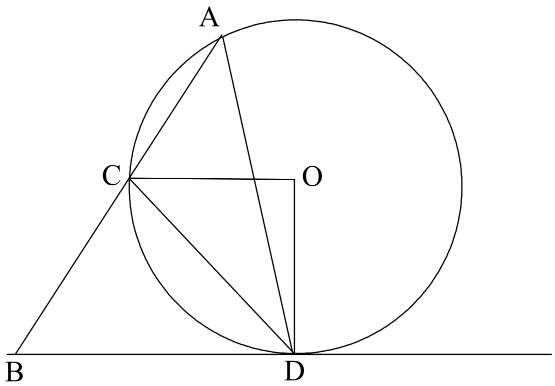
Therefore $B\hat{D}C = \frac{\pi}{2} - C\hat{D}O = \frac{\pi}{5}$ so that

$C\hat{D}A = B\hat{D}A - B\hat{D}C = \frac{\pi}{5} = D\hat{A}B$

so that the triangle CDA is isosceles and consequently $AC = DC$. ***(A1)***

Finally $D\hat{C}B = \pi - B\hat{D}C - A\hat{B}D = \frac{2\pi}{5} = C\hat{B}D$ so that the triangle BDC is isosceles and therefore $DC = DB$ ***(A1)*** ***(A1)***

i.e. $AC = DC = DB$ ***(AG)***



[5 marks]

Total [30 marks]