INTERNATIONAL BACCALAUREATE
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MARKSCHEME

November 2001

MATHEMATICS

Higher Level

Paper 1

13 pages

1. $n = 1800,\ p = \frac{2}{3}$

(a) $\mathrm{E}(X) = np = 1200$ *(A1)* *(C1)*

(b) $\mathrm{SD}(X) = \sqrt{np(1-p)} = \sqrt{1200 \times \frac{1}{3}} = 20$ *(M1)(A1)* *(C2)*

[3 marks]

2. $\mathrm{i}(z+2) = 1-2z \quad \Rightarrow \quad (2+\mathrm{i})z = 1-2\mathrm{i}$

$\Rightarrow \quad z = \frac{1-2\mathrm{i}}{2+\mathrm{i}}$ *(M1)*

$= \frac{1-2\mathrm{i}}{2+\mathrm{i}} \times \frac{2-\mathrm{i}}{2-\mathrm{i}}$ *(M1)*

$= \frac{-5\mathrm{i}}{5}$

$= -\mathrm{i}$. *(A1)* *(C3)*

$(a = 0,\ b = -1)$

[3 marks]

3. The remainder when divided by $(x-2)$ is $f(2) = 8 + 12 + 2a + b = 2a + b + 20$ *(M1)*
and when divided by $(x+1)$, the remainder is $f(-1) = -1 + 3 - a + b = 2 - a + b$. *(M1)*
These remainders are equal when $2a + 20 = 2 - a$
giving $a = -6$. *(A1)* *(C3)*

[3 marks]

4. (a) The series converges provided $-1 < \frac{2x}{3} < 1$. *(M1)*

This gives $-1.5 < x < 1.5$ or $|x| < \frac{3}{2}$ *(A1)* *(C2)*

(b) When $x = 1.2$, the common ratio is $r = 0.8$ and the sum is $\frac{1}{1-0.8} = 5$ *(A1)* *(C1)*

[3 marks]

5. Let $x = \frac{2y+1}{y-1}$ *(M1)*

$\Rightarrow \quad xy - x = 2y + 1$

$\Rightarrow \quad y(x-2) = x + 1$

Therefore, $f^{-1} : x \mapsto \frac{x+1}{x-2}$, *(A1)* *(C2)*

Domain $x \in \mathbb{R}, x \neq 2$ *(A1)* *(C1)*

[3 marks]

6. $\boldsymbol{AB} = \begin{pmatrix} x & 4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & y \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} 2x+32 & xy+16 \\ 24 & 4y+8 \end{pmatrix}$ *(A1)*

$\boldsymbol{BA} = \begin{pmatrix} 2 & y \\ 8 & 4 \end{pmatrix}\begin{pmatrix} x & 4 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 2x+4y & 2y+8 \\ 8x+16 & 40 \end{pmatrix}$ *(A1)*

$\boldsymbol{AB} = \boldsymbol{BA} \quad \Rightarrow \quad 8x+16=24$ and $4y+8=40$

This gives $x=1$ and $y=8$. *(A1)* *(C3)*

[3 marks]

7. For the curve, $y=7$ when $x=1$ $\Rightarrow$ $a+b=14$, and *(M1)*

$\frac{dy}{dx} = 6x^2+2ax+b=16$ when $x=1$ $\Rightarrow$ $2a+b=10$. *(M1)*

Solving gives $a=-4$ and $b=18$. *(A1)* *(C3)*

[3 marks]

8. **METHOD 1**

$\mathrm{E}(X) = \int_0^1 \frac{4x}{\pi(1+x^2)}\,dx$ *(M1)*

$= 0.441$. *(G2)* *(C3)*

METHOD 2

$\mathrm{E}(X) = \int_0^1 \frac{4x}{\pi(1+x^2)}\,dx$ *(M1)*

$= \frac{2}{\pi}\left[\ln(1+x^2)\right]_0^1$ *(M1)*

$= \frac{2}{\pi}(\ln 2) \quad \left[\text{or } \frac{\ln 4}{\pi}\right]$. *(A1)* *(C3)*

[3 marks]

9. The matrix is singular if its determinant is zero. *(M1)*

Then, $\begin{vmatrix} 1 & -2 & -3 \\ 1 & -k & -13 \\ -3 & 5 & k \end{vmatrix} = \begin{vmatrix} -k & -13 \\ 5 & k \end{vmatrix} + 2\begin{vmatrix} 1 & -13 \\ -3 & k \end{vmatrix} - 3\begin{vmatrix} 1 & -k \\ -3 & 5 \end{vmatrix}$

$= -k^2+65+2k-78-15+9k$

$= -(k^2-11k+28)$ *(A1)*

$= -(k-4)(k-7)$.

Therefore, the matrix is singular if $k=4$ or $k=7$. *(A1)* *(C3)*

[3 marks]

10. (a) $\frac{dy}{dx} = \sec^2 x - 8\cos x$ *(A1)* *(C1)*

(b) $\frac{dy}{dx} = \frac{1-8\cos^3 x}{\cos^2 x}$ *(M1)*

$\frac{dy}{dx} = 0$

$\Rightarrow \cos x = \frac{1}{2}$ *(A1)* *(C2)*

[3 marks]

11. **METHOD 1**

$|5-3x| \le |x+1|$

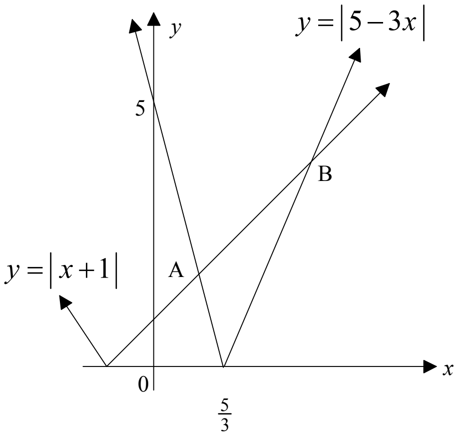
$\Rightarrow \quad 25 - 30x + 9x^2 \le x^2 + 2x + 1$ *(M1)*

$\Rightarrow \quad 8x^2 - 32x + 24 \le 0$

$\Rightarrow \quad 8(x-1)(x-3) \le 0$ *(M1)*

$\Rightarrow \quad 1 \le x \le 3$ *(A1)* *(C3)*

METHOD 2



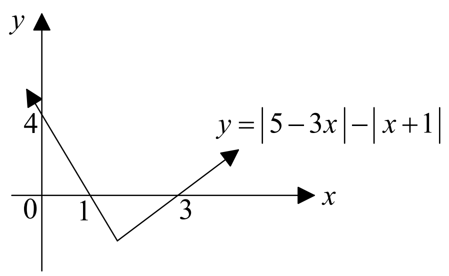
(G1)

We obtain A = (1, 2) and B = (3, 4) *(G1)*

Therefore, $1 \le x \le 3$. *(A1)* *(C3)*

METHOD 3

Sketch the graph of $y = |5-3x| - |x+1|$.



(G2)

From this graph we see that $y \le 0$ for $1 \le x \le 3$. *(A1)* *(C3)*

[3 marks]

12. The uppermost vertex of triangle 2 has coordinates $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. *(A1)*

Either $(0, 0) \mapsto (0, 0), (1, 0) \mapsto (1, 0)$ and $(0, 1) \mapsto \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, or

$(0, 0) \mapsto (0, 0), (1, 0) \mapsto \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $(0, 1) \mapsto (1, 0)$ *(M1)*

Therefore, a suitable matrix is either $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$ or $\begin{pmatrix} \frac{1}{2} & 1 \\ \frac{\sqrt{3}}{2} & 0 \end{pmatrix}$. *(A1)* *(C3)*

[3 marks]

13. **METHOD 1**

(a) The equation of the tangent is $y = -4x - 8$. *(G2)* *(C2)*

(b) The point where the tangent meets the curve again is $(-2, 0)$. *(G1)* *(C1)*

METHOD 2

(a) $y = -4$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 8x + 1 = -4$ at $x = -1$. *(M1)*
Therefore, the tangent equation is $y = -4x - 8$. *(A1)* *(C2)*

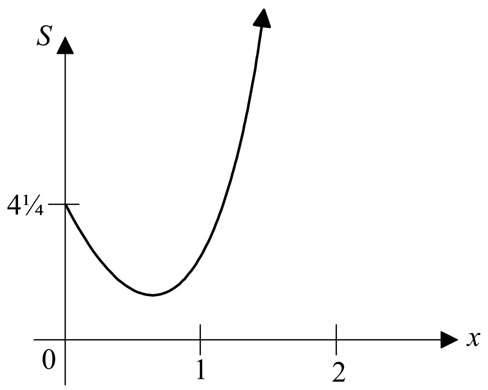
(b) This tangent meets the curve when $-4x - 8 = x^3 + 4x^2 + x - 6$ which gives
$x^3 + 4x^2 + 5x + 2 = 0 \Rightarrow (x+1)^2(x+2) = 0$.
The required point of intersection is $(-2, 0)$. *(A1)* *(C1)*

[3 marks]

14. **METHOD 1**

Let $S = \mathrm{AP}^2 = (x-2)^2 + \left(x^2 + \frac{1}{2}\right)^2$. *(M1)*

The graph of S is as follows:



The minimum value of S is 2.6686. *(G1)*

Therefore the minimum distance $= \sqrt{2.6686} = 1.63$ (3 s.f.) *(A1)*

OR

The minimum point is (0.682, 1.63) *(G1)*

The minimum distance is 1.63 (3 s.f.) *(G1)* *(C3)*

METHOD 2

Let $S = \mathrm{AP}^2 = (x-2)^2 + \left(x^2 + \frac{1}{2}\right)^2$. *(M1)*

$$\frac{\mathrm{d}S}{\mathrm{d}x} = 2(x-2) + 4x\left(x^2 + \tfrac{1}{2}\right) = 4(x^3 + x^2 - 1)$$

Solving $x^3 + x - 1 = 0$ gives $x = 0.68233$ *(G1)*

Therefore the minimum distance $= \sqrt{(0.68233-2)^2 + (0.68233^2 + 0.5)^2} = 1.63$ (3 s.f.) *(A1)* *(C3)*

[3 marks]

15. The direction of the line is $\boldsymbol{v} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $|\boldsymbol{v}| = 3$. *(A1)*

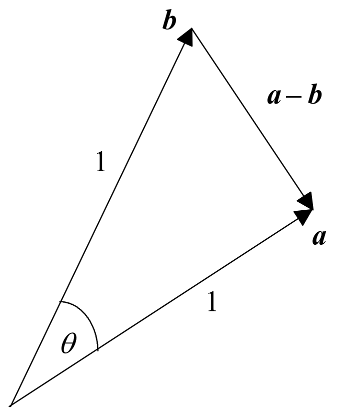
Therefore, the position vector of any point on the line 6 units from A is

$3\boldsymbol{i} - 2\boldsymbol{k} \pm 2\boldsymbol{v} = 7\boldsymbol{i} - 4\boldsymbol{j}$ or $-\boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k}$, *(M1)*

giving the point $(7, -4, 0)$ or $(-1, 4, -4)$. *(A1)* *(C3)*

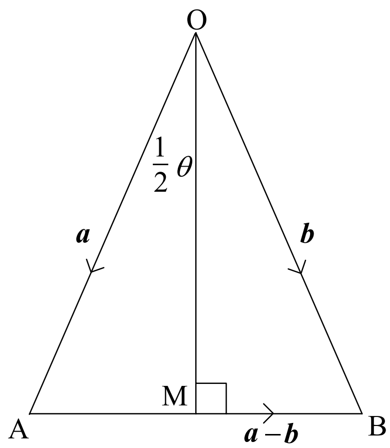
[3 marks]

16. **METHOD 1**



$$\left|\boldsymbol{a}-\boldsymbol{b}\right| = \sqrt{1^2+1^2-2(1)(1)\cos\theta} \quad \textit{(M1)}$$
$$= \sqrt{2(1-\cos\theta)} \quad \textit{(A1)}$$
$$= \sqrt{4\sin^2\frac{1}{2}\theta}$$
$$= 2\sin\frac{1}{2}\theta\,. \quad \textit{(A1)} \quad \textit{(C3)}$$

METHOD 2



$\overrightarrow{\mathrm{AB}} = \boldsymbol{a}-\boldsymbol{b}$

In ΔOAM, $\mathrm{AM} = \mathrm{OA}\sin\frac{1}{2}\theta$. ***(M1)(A1)***

Therefore, $\left|\boldsymbol{a}-\boldsymbol{b}\right| = 2\sin\frac{1}{2}\theta$. ***(A1)*** ***(C3)***

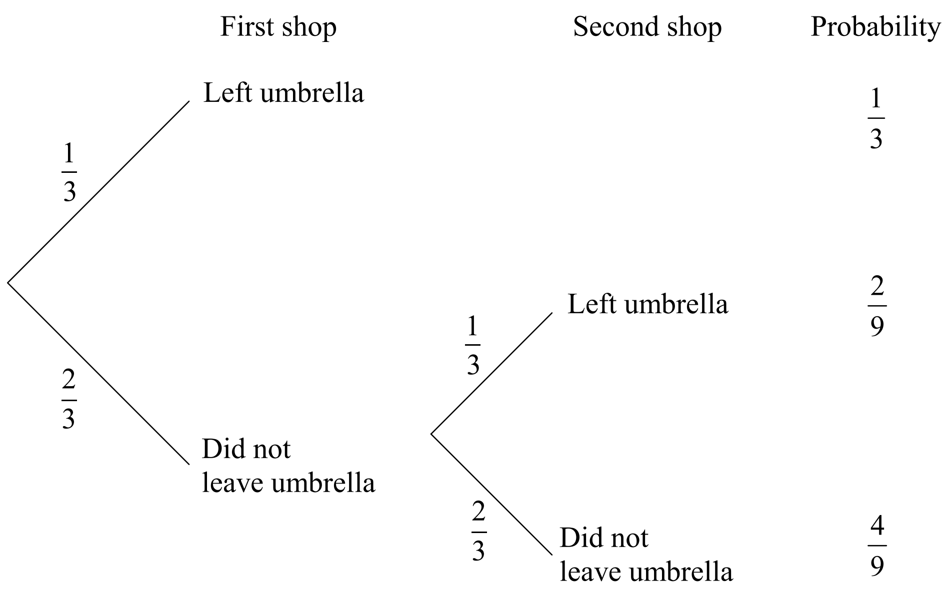
[3 marks]

17. The total number of four-digit numbers $= 9\times10\times10\times10 = 9000$. ***(A1)***
The number of four-digit numbers which **do not** contain a digit 3
$= 8\times9\times9\times9 = 5832$. ***(A1)***

Thus, the number of four-digit numbers which contain at least one digit 3 is
$9000-5832 = 3168$. ***(A1)*** ***(C3)***

[3 marks]

18.



(M1)(A1)

Required probability $=\dfrac{\frac{2}{9}}{\frac{2}{9}+\frac{1}{3}}=\dfrac{2}{5}$. ***(A1)*** ***(C3)***

[3 marks]

19. If A g is present at any time, then $\dfrac{\mathrm{d}A}{\mathrm{d}t}=kA$ where k is a constant.

Then, $\displaystyle\int\frac{\mathrm{d}A}{A}=k\int\mathrm{d}t$

$\Rightarrow \quad \ln A = kt + c$

$\Rightarrow \quad A = \mathrm{e}^{kt+c} = c_1\mathrm{e}^{kt}$

When $t=0$, $c_1=50$, $\Rightarrow 48=50\mathrm{e}^{10k}$. ***(A1)***

$\dfrac{\ln 0.96}{10}=k$ or $k=-0.00408(2)$ ***(A1)***

For half life, $25=50\mathrm{e}^{kt}$

$\Rightarrow \quad \ln 0.5 = kt$

$\Rightarrow \quad t=\dfrac{10\ln 0.5}{\ln 0.96}=169.8$.

Therefore, half-life = 170 years (3 s.f.) ***(A1)*** ***(C3)***

[3 marks]

20. The curves meet when $x=-1.5247$ and $x=0.74757$. ***(G1)***

The required area $=\displaystyle\int_{-1.5247}^{0.74757}\left(\frac{2}{1+x^2}-\mathrm{e}^{\frac{x}{3}}\right)\mathrm{d}x$ ***(M1)***

$=1.22$. ***(G1)*** ***(C3)***

[3 marks]