INTERNATIONAL BACCALAUREATE
BACCALAURÉAT INTERNATIONAL
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MARKSCHEME

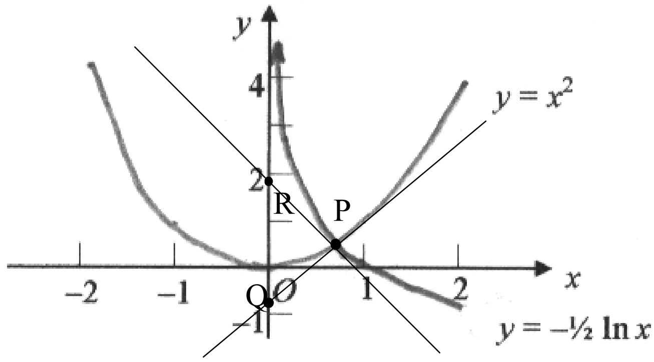
November 2000

MATHEMATICS

Higher Level

Paper 2

1. (a)



(C2)

Note: Award ***(C1)*** for $y = x^2$, ***(C1)*** for $y = -\frac{1}{2}\ln x$

[2 marks]

(b) $x^2 + \frac{1}{2}\ln x = 0$ when $x = 0.548217$.

Therefore, the x-coordinate of P is 0.548…. ***(G2)***

[2 marks]

(c) The tangent at P to $y = x^2$ has equation $y = 1.0964x - 0.30054$, ***(G2)***

and the tangent at P to $y = -\frac{1}{2}\ln x$ has equation $y = -0.91205x + 0.80054$. ***(G2)***

Thus, the area of triangle PQR $= \frac{1}{2}(0.30052 + 0.80054)(0.5482)$. ***(M1)***

$= 0.302$ (3 s.f.) ***(A1)***

OR

$y = x^2 \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x$ ***(M1)***

Therefore, the tangent at (p, p^2) has equation $2px - y = p^2$. ***(C1)***

$y = -\frac{1}{2}\ln x \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{2x}$ ***(M1)***

Therefore, the tangent at (p, p^2) has equation $x + 2py = p + 2p^3$. ***(C1)***

Thus, Q $= (0, -p^2)$ and R $= (0, p^2 + \frac{1}{2})$.

Thus, the area of the triangle PQR

$= \frac{1}{2}(2p^2 + \frac{1}{2})p$ ***(M1)***

$= 0.302$ (3 s.f.) ***(A1)***

[6 marks]

(d) $y = x^2 \Rightarrow$ when $x = a$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2a$ ***(C1)***

$y = -\frac{1}{2}\ln x \Rightarrow$ when $x = a$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{2a}$ $(a > 0)$ ***(C1)***

Now, $(2a)\left(-\dfrac{1}{2a}\right) = -1$ for all $a > 0$. ***(M1)***

Therefore, the tangents to the curve at $x = a$ on each curve are always perpendicular. ***(R1)(AG)***

[4 marks]

[Total: 14 marks]

2. (a) (i) $\dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{1}{125l^3}(x^2 - lx)$ ***(M1)(AG)***

(ii) $w(x) = \int z(x)\,\mathrm{d}x + C = \dfrac{1}{125l^3}\left(\dfrac{x^4}{12} - \dfrac{lx^3}{6}\right) + \dfrac{x}{1500} + C$ ***(M1)(A1)***

Hence, $C = w(0) = 0$ ***(A1)***

and therefore, $w(x) = \dfrac{1}{125l^3}\left(\dfrac{x^4}{12} - \dfrac{lx^3}{6}\right) + \dfrac{x}{1500}$ ***(A1)***

(iii) $\dfrac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{1}{125l^3}(x^2 - lx)$ ***(A1)***

We have seen above that $w(0) = 0$

$w(l) = \dfrac{1}{125l^3}\left(\dfrac{l^4}{12} - \dfrac{l^4}{6}\right) + \dfrac{l}{1500} = -\dfrac{l}{1500} + \dfrac{l}{1500} = 0$ ***(A2)***

[8 marks]

(b) When $l = 2.4, x = 1.2$ at the centre of the rod.

Now, $y(1.2) = \dfrac{1}{125(2.4)^3}\left(\dfrac{1.2^4}{12} - \dfrac{2.4(1.2)^3}{6}\right) + \dfrac{1.2}{1500}$ ***(M1)***

$= 0.0005 \text{ m}$. ***(A1)***

[2 marks]

[Total: 10 marks]

3. (i) (a) P(all ten cells fail) $= 0.8^{10} = 0.107$. ***(M1)(A1)***

[2 marks]

(b) P(satellite is still operating at the end of one year)
$= 1 - \mathrm{P}$(all ten cells fail within one year) ***(M1)***
$= 1 - 0.107$
$= 0.893$. ***(A1)***

[2 marks]

(c) P(satellite is still operating at the end of one year)
$= 1 - 0.8^n$. ***(C1)***

We require the smallest n for which $1 - 0.8^n \geq 0.95$. ***(M1)***
Thus, $0.8^n \leq 0.05$

$$\left(\frac{5}{4}\right)^n \geq 20$$

$$n \geq \frac{\log 20}{\log 1.25} = 13.4$$ ***(M1)(A1)***

Therefore, 14 solar cells are needed. ***(C1)***

[5 marks]

(ii) (a) Required probability
$= \mathrm{P}(Y \leq \tfrac{1}{2})$
$= \int_0^{1/2} 0.5\mathrm{e}^{-y/2}\mathrm{d}y$ ***(M2)***
$= 0.2212$. ***(G1)***

OR

Required probability $= \int_0^{1/2} 0.5\mathrm{e}^{-y/2}\mathrm{d}y$ ***(M1)***
$= -\left[\mathrm{e}^{-y/2}\right]_0^{1/2}$ ***(M1)***
$= 1 - \mathrm{e}^{-1/4}$
$= 0.2212$ (4 s.f.) ***(A1)***

[3 marks]

(b) Required probability
$=$ P(2 or 3 of the components fail in six months) ***(M1)***
$= \binom{3}{2}(0.2212)^2(0.7788) + (0.2212)^3$ ***(M2)***
$= 0.125$. ***(G1)***

[4 marks]

[Total: 16 marks]

4. (i) (a) Since det $\boldsymbol{A} \neq 0$, $\boldsymbol{A}^{-1}$ exists. *(M1)*

Hence $\boldsymbol{AB} = \boldsymbol{C} \Rightarrow \boldsymbol{B} = \boldsymbol{A}^{-1}\boldsymbol{C}$ *(C1)*

[2 marks]

(b) (i) $$\boldsymbol{DA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ *(A1)*

(ii) $\boldsymbol{B} = \boldsymbol{A}^{-1}\boldsymbol{C} = \boldsymbol{DC}$ *(M1)*

$$= \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$ *(A1)*

[3 marks]

(c) The system of equations is
$$\begin{aligned} x + 2y + 3z &= 5 \\ 2x - y + 2z &= 7 \\ 3x - 3y + 2z &= 10 \end{aligned}$$

or $\boldsymbol{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{C}$. *(M1)*

The required point $= (1, -1, 2)$. *(A1)*

[2 marks]

(ii) (a) $$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = 7\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}.$$ *(M2)(AG)*

[2 marks]

(b) $$\boldsymbol{w} = \begin{pmatrix} \lambda + 2\mu \\ 2\lambda - \mu \\ 3\lambda + 2\mu \end{pmatrix}$$ *(C1)*

The line of intersection of the planes is parallel to $\boldsymbol{u} \times \boldsymbol{v}$. *(M1)*
Now, $\boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = 7\lambda + 14\mu + 8\lambda - 4\mu - 15\lambda - 10\mu = 0$ for all λ, μ. *(M1)(C1)*
Therefore, $\boldsymbol{w}$ is perpendicular to the line of intersection of the given planes. *(AG)*

OR

The line of intersection of the planes is perpendicular to $\boldsymbol{u}$ and to $\boldsymbol{v}$, so it will be perpendicular to the plane containing $\boldsymbol{u}$ and $\boldsymbol{v}$, that is, to all vectors of the form $\lambda\boldsymbol{u} + \mu\boldsymbol{v} = \boldsymbol{w}$. *(M2)* *(R1)* *(C1)*

[4 marks]

[Total: 13 marks]

5. (a) (i) $y = \dfrac{a + b\sin x}{b + a\sin x}$, $0 < a < b$

$$\frac{dy}{dx} = \frac{(b + a\sin x)(b\cos x) - (a + b\sin x)(a\cos x)}{(b + a\sin x)^2}$$ ***(M1)(C1)***

$$= \frac{b^2\cos x + ab\sin x\cos x - a^2\cos x - ab\sin x\cos x}{(b + a\sin x)^2}$$ ***(M1)(C1)***

$$= \frac{(b^2 - a^2)\cos x}{(b + a\sin x)^2}$$ ***(AG)***

[4 marks]

(ii) $\dfrac{dy}{dx} = 0 \Rightarrow \cos x = 0$ since $b^2 - a^2 \neq 0$.

This gives $x = \dfrac{\pi}{2}(+\pi k,\ k \in \mathbb{Z})$ ***(M1)(C1)***

When $x = \dfrac{\pi}{2}$, $y = \dfrac{a+b}{b+a} = 1$, and when $x = \dfrac{3\pi}{2}$, $y = \dfrac{a-b}{b-a} = -1$.

Therefore, maximum $y = 1$ and minimum $y = -1$. ***(A2)***

[4 marks]

(iii) A vertical asymptote at the point x exists if and only if $b + a\sin x = 0$. ***(R1)***

Then, since $0 < a < b$, $\sin x = -\dfrac{b}{a} < -1$, which is impossible. ***(R1)***

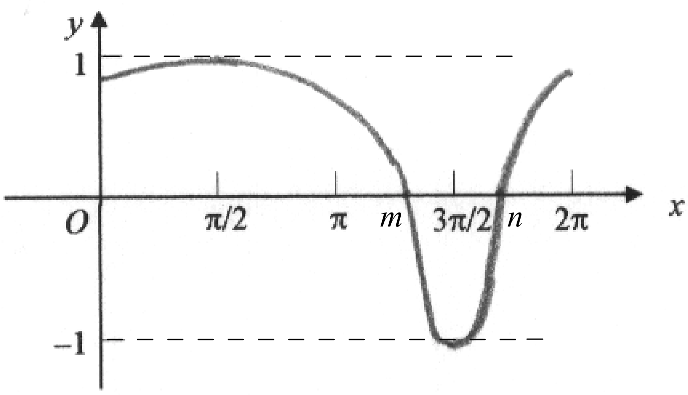
Therefore, no vertical asymptote exists. ***(AG)***

[2 marks]

(b) (i) y-intercept $= 0.8$ ***(A1)***

(ii) For x-intercepts, $\sin x = -\dfrac{4}{5} \Rightarrow x = 4.069, 5.356$. ***(A2)***

(iii)



(C2)

[5 marks]

(c) Area $= \displaystyle\int_0^{4.069} \frac{4 + 5\sin x}{5 + 4\sin x}\,dx - \int_{4.069}^{5.356} \frac{4 + 5\sin x}{5 + 4\sin x}\,dx$ ***(M1)(C1)***

OR

Area $= \displaystyle\int_0^{5.356} \left|\frac{4 + 5\sin x}{5 + 4\sin x}\right| dx$ ***(M1)(C1)***

[2 marks]

[Total: 17 marks]

6. (i) (a) Let X be the random variable representing the length of the rod.

X is $\mathrm{N}(100, 15^2)$

$\mathrm{P}(X > 105) = 1 - 0.6306$ ***(M1)***

$= 0.369$ (3 s.f.) ***(A1)***

[2 marks]

(b) $\bar{X}$ is $\mathrm{N}\left(100, \dfrac{15^2}{60}\right)$ ***(M1)***

(M1)

$\mathrm{P}(\bar{X} > 105) = 1 - 0.9951$

$= 0.0049$ ***(A1)***

[3 marks]

(ii) This is a χ^2-test for independence between two variables. The expected frequency in each cell is calculated by $\dfrac{\text{row total} \times \text{column total}}{\text{grand total}}$. The expected frequencies are given below ***(R2)***

	82.726	17.688	22.586
	51.115	10.929	13.956
	109.63	23.44	29.931
	60.531	12.942	16.527

(G3)

H_0 : There is no association between alcohol and nicotine consumption.

H_1 : There is some association. ***(A1)***

The critical number with 6 degrees of freedom and 5 % level of significance is 12.5916. ***(A1)***

The test statistic is $\chi^2_{calc} = \sum \dfrac{(f_e - f_o)^2}{f_e} = 42.252$. ***(G1)***

Since 42.252 > 12.5916, we reject the null hypothesis and conclude that we have evidence that there is some association between nicotine and alcohol consumption. ***(R1)***

[9 marks]

continued...

Question 6 continued

(iii) (a) This is a t-test of the difference of two means. Our assumptions are that the two populations are approximately normal, samples are random, and they are independent from each other. ***(R1)***

$H_0 : \mu_1 - \mu_2 = 0$
$H_1 : \mu_1 - \mu_2 < 0$ ***(A1)***

$t = -2.460$, ***(A1)***
degrees of freedom $= 38$ ***(A1)***
Since the value of critical $t = -1.686$ we reject H_0. ***(A1)***
Hence group B grows faster. ***(R1)***

[6 marks]

(b) This is a χ^2 goodness-of-fit test.
To finish the table, the frequencies of the respective cells have to be calculated. Since the standard deviation is not given, it has to be estimated using the data itself. $s = 49.59$,
e.g. the third expected frequency is $40 \times 0.308 = 12.32$,
since $P(350.5 < W < 390.5) = 0.3078\ldots$
The table of observed and expected frequencies is:

Amount of weight gain	Observed	Expected
271 – 310	2	3.22
311 – 350	9	7.82
351 – 390	8	12.32
391 – 430	15	10.48
431 – 470	6	6.17

(M1)(A2)

Since the first expected frequency is 3.22, we combine the two cells, so that the first two rows become one row, that is,

271 – 350	11	11.04

(M1)

Number of degrees of freedom is $4 - 1 - 1 = 2$ ***(C1)***

H_0 : The distribution is normal with mean 380
H_1 : The distribution is not normal with mean 380 ***(A1)***

The test statistic is

$$\chi^2_{calc} = \sum \frac{(f_e - f_o)^2}{f_e} = \frac{(11-11.04)^2}{11.04} + \frac{(8-12.32)^2}{12.32} + \frac{(15-10.48)^2}{10.48} + \frac{(6-6.17)^2}{6.17}$$

$= 3.469$ ***(A1)***

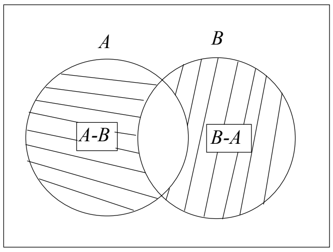
With 2 degrees of freedom, the critical number is $\chi^2 = 5.99$ ***(A2)***

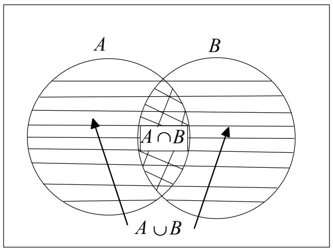
So, we do not have enough evidence to reject the null hypothesis. Therefore, there is no evidence to say that the distribution is not normal with mean 380. ***(R1)***

[10 marks]

[Total: 30 marks]

7. (i) (a)





(A1)(A1)

[2 marks]

(b) $(A \cup B) - (B \cap A) = (A \cup B) \cap (B \cap A)'$

$= [A \cap (B \cap A)'] \cup [B \cap (B \cap A)']$ ***(A1)***

$= [A \cap (B' \cup A')] \cup [B \cap (B' \cup A')]$ ***(M1)***

$= (A \cap B') \cup (A \cap A') \cup (B \cap B') \cup (B \cap A') = (A \cap B') \cup (B \cap A')$ ***(M1)***

$= (A - B) \cup (B - A)$ ***(A1)***

[4 marks]

(ii) Let $\boldsymbol{X} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ and $\boldsymbol{Y} = \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}$. Then $\boldsymbol{XY} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ac & ad+b \\ 0 & 1 \end{pmatrix}$. ***(M1)***

Since $ac = \pm 1$ and $ad + b \in \mathbb{Z}$, then $\boldsymbol{XY} \in H$ ***(R1)***

Since matrix multiplication is associative, so is the operation in this case. ***(A1)***

Since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is of the required form, it is an element of *H*, and so the set has an identity element under this operation. ***(R1)***

Let $\boldsymbol{X} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Since $\det(\boldsymbol{X}) = a \neq 0$, then $\boldsymbol{X}^{-1}$ exists for every element of *H*.

$\boldsymbol{X}^{-1} = \frac{1}{a}\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & -\frac{b}{a} \\ 0 & 1 \end{pmatrix} \in H$ ***(M1)***

$\left(\text{since } a = \pm 1, \frac{1}{a} = \pm 1 = a, \text{ and } \frac{-b}{a} = \pm b \in \mathbb{Z} \right)$ ***(R1)***

[6 marks]

(iii) (a) If *G* is a group and *H* is a subgroup of *G* then the order of *H* is a divisor of the order of *G*. ***(A2)***

[2 marks]

(b) Since the order of *G* is 24, the order of *a* must be 1, 2, 3, 4, 6, 8, 12 or 24 ***(R2)***

The order cannot be 1, 2, 3, 6 or 12 since $a^{12} \neq e$ ***(R1)***

Also $a^8 \neq e$ so that the order of *a* must be 24 ***(R1)***

Therefore, *a* is a generator of *G*, which must therefore be cyclic. ***(R1)***

[5 marks]

continued…

Question 7 continued

(iv) (a) Since $(a+b\sqrt{2})(c+d\sqrt{2})=ac+2bd+(ad+bc)\sqrt{2}$,
and $(ac+2bd)^2-2(ad+bc)^2=(a^2-2b^2)(c^2-2d^2)\neq 0$,
S is closed under multiplication. ***(A2)***
$1=1+0\sqrt{2}$ is the neutral element. ***(A1)***
Finally, $\dfrac{a-b\sqrt{2}}{a^2-2b^2}\in S$ ***(M1)***
and $\left(\dfrac{a-b\sqrt{2}}{a^2-2b^2}\right)(a+b\sqrt{2})=1$, so every element of S has an inverse. ***(A1)***

[5 marks]

(b) To show that $f(x)$ is an isomorphism, we need to show that it is injective, surjective and that it preserves the operation.
Injection: Let $x_1=a+b\sqrt{2}$, $x_2=c+d\sqrt{2}$
$f(x_1)=f(x_2)\Rightarrow a-b\sqrt{2}=c-d\sqrt{2}\Rightarrow(a-c)+(d-b)\sqrt{2}=0$ ***(M1)***
$\Rightarrow a=c$, and $b=d\Rightarrow x_1=x_2$ ***(A1)***
Surjection: For every $y=a-b\sqrt{2}$ there is $x=a+b\sqrt{2}$ ***(M1)(A1)***
Preserves operation:
$f(x_1x_2)=f\left(\left(a+b\sqrt{2}\right)\left(c+d\sqrt{2}\right)\right)=f\left(ac+2bd+(ad+bc)\sqrt{2}\right)$ ***(M1)***
$=ac+2bd-(ad+bc)\sqrt{2}=\left(a-b\sqrt{2}\right)\left(c-d\sqrt{2}\right)$ ***(M1)***
$\left(f\left(a+b\sqrt{2}\right)\right)\left(f\left(c+d\sqrt{2}\right)\right)=\left(f(x_1)\right)\left(f(x_2)\right)$

[6 marks]

[Total: 30 marks]

8. (i) (a) $c_1 = -1, c_2 = 8, c_3 = -48,$ ***(A1)***
$c_4 = 256, c_5 = -1280$ ***(A1)***
[2 marks]

(b) The characteristic polynomial is $x^2 + 8x + 16 = 0$ ***(A1)***
Its solution is $x = -4$ ***(A1)***
[2 marks]

(c) Since there is only one solution to the characteristic polynomial, the general solution is of the form: $c_n = u(-4)^n + v\,n(-4)^n$, with u and v to be determined. ***(R1)***

Since $c_1 = -1 \Rightarrow -4u - 4v = -1$
$c_2 = 8 \Rightarrow 16u + 32v = 8$ ***(M1)***
$\Rightarrow u = 0$, and $v = 1/4$ ***(A1)***
Therefore $c_n = (1/4)n(-4)^n$ ***(A1)***
[4 marks]

(ii) (a) $(8n+3) = (5n+2) + (3n+1)$
$(5n+2) = (3n+1) + (2n+1)$
$(3n+1) = (2n+1) + n$
$(2n+1) = 2n+1$ ***(M2)***
The greatest common divisor of $(8n+3)$ and $(5n+2)$ is 1, and hence they are relatively prime. ***(M1)(R1)***
[4 marks]

(b) (i) $10 \equiv 1 \bmod 3 \Rightarrow 10^n \equiv 1 \bmod 3$ ***(M1)(A1)***
$\Rightarrow (10^n r_n + 10^{n-1} r_{n-1} + \ldots + r_0) \equiv (r_n + r_{n-1} + \ldots + r_0) \bmod 3$ ***(M1)(AG)***
[3 marks]

(ii) From the previous result, $(10^n r_n + 10^{n-1} r_{n-1} + \ldots + r_0)$ and $(r_n + r_{n-1} + \ldots + r_0)$ have the same remainder when divided by 3. ***(M1)(A1)***
$\Rightarrow$ if 3 divides $(r_n + r_{n-1} + \ldots + r_0)$ then it divides a. ***(C1)***
$1 + 3 + 7 + \ldots + x = 3k, k \in \mathbb{Z}$
$\Rightarrow (38 + x) \equiv 0 \bmod 3, \Rightarrow (2 + x) \equiv 0 \bmod 3$ ***(M1)***
$\Rightarrow x = 1, 4$, or 7 ***(A2)***
[6 marks]

(iii) (a) Since every face is enclosed by at least c edges, Euler's theorem: $f = 2 + e - v$ ***(M1)***
$2e \geq cf$ ***(A1)***
$\Rightarrow 2e \geq c(2 + e - v) \Rightarrow 2e \geq 2c + ec - cv$ ***(M2)***
$\Rightarrow e(c-2) \leq c(v-2)$ ***(A1)***
$\Rightarrow e \leq \dfrac{c(v-2)}{c-2}$ ***(AG)***
[5 marks]

(b) In $\kappa_{3,3}$ the minimum length c is 4 ***(A1)***
$9 \leq \dfrac{4(6-2)}{4-2} = 8$, which is a contradiction ***(M2)***
Hence the graph cannot be planar. ***(R1)***
[4 marks]

[Total: 30 marks]

9. (a) (i) Since $e^{kx} \neq 0$ for any value of $x \in \mathbb{R}$, the only zeros possible are those of $x^2 - 1 = 0$. ***(R1)***

This equation clearly has two solutions $x = \pm 1$ ***(A1)***

[2 marks]

(ii) $f'(x) = ke^{kx}(x^2 - 1) + 2xe^{kx} = e^{kx}(kx^2 + 2x - k)$ ***(M1)(A1)***

$f'(x) = 0$ if $kx^2 + 2x - k = 0$

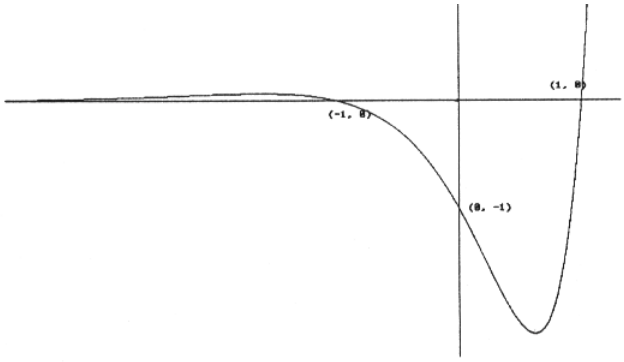
$\Rightarrow x = \dfrac{-2 \pm \sqrt{4 + 4k^2}}{2k} = \dfrac{1}{k}\left(-1 \pm \sqrt{1 + k^2}\right)$ ***(M1)(A1)***

$\Rightarrow x_1 x_2 = \dfrac{1}{k}\left(-1 + \sqrt{1 + k^2}\right) \cdot \dfrac{1}{k}\left(-1 - \sqrt{1 + k^2}\right)$ ***(M1)***

$\Rightarrow x_1 x_2 = \dfrac{1}{k^2}\left(+1 - 1 - k^2\right) = -1$ ***(AG)***

[5 marks]

(b) (i) $f(x) = (x^2 - 1)e^{2x}$



(C1)(A1)

Note: Award ***(C1)*** for graph, and ***(A1)*** for the intercepts.

$f'(x) = e^{2x}\left(2x^2 + 2x - 2\right)$ ***(M1)***

$\Rightarrow f'(x) = 0 \Rightarrow x = \frac{1}{2}\left(-1 \pm \sqrt{5}\right)$

$\Rightarrow$ maximum at $\frac{1}{2}\left(-1 - \sqrt{5}\right)$ is $\frac{1}{2}\left(1 + \sqrt{5}\right)e^{-1-\sqrt{5}}$

$\Rightarrow$ minimum at $\frac{1}{2}\left(-1 + \sqrt{5}\right)$ is $\frac{1}{2}\left(1 - \sqrt{5}\right)e^{-1+\sqrt{5}}$ ***(A1)***

[4 marks]

(ii) $(x^2 - 1)e^{2x} - e = 0$. Let $h(x) = (x^2 - 1)e^{2x} - e$

$h'(x) = (2x^2 + 2x - 2)e^{2x}$ ***(M1)***

$x_{n-1} = x_n - \dfrac{\left(x_n^2 - 1\right)e^{2x_n} - e}{\left(2x_n^2 + 2x_n - 2\right)e^{2x_n}}$ ***(A1)***

$x_1 = 1 - \dfrac{-e}{2e^2} = 1.183939\ldots$

$x_2 = 1.18394 - \dfrac{h(1.18394)}{h(1.18394)} = 1.375654\ldots$

$x_3 = 1.132445\ldots$ ***(A1)***

$x_4 = 1.132387\ldots = 1.13239$ (5 d.p.) ***(A1)***

$x_5 = 1.132387\ldots = 1.13239$ (5 d.p.) ***(A1)***

Since $x_4 = x_5$ to 5 d.p., $x = 1.13239$ (5 d.p.) ***(R1)***

[6 marks]

continued...

Question 9 (b) continued

(iii) $h(x) = \mathrm{e}^{kx}$, $h(0) = 1$

$h'(x) = k\mathrm{e}^{kx}$, $h'(0) = k$ ***(M1)***

$h''(x) = k^2\mathrm{e}^{kx}$, $h''(0) = k^2$ ***(M1)***

$h'''(x) = k^3\mathrm{e}^{kx}$, $h'''(0) = k^3$ ***(M1)***

$P(x) = 1 + kx + \dfrac{k^2x^2}{2} + \dfrac{k^3x^3}{6}$ ***(C1)***

[4 marks]

(iv) $\left(1 + 2x + 2x^2 + \dfrac{4x^3}{3}\right)\left(x^2 - 1\right) =$

$\dfrac{4x^5}{3} + 2x^4 + \dfrac{2x^3}{3} - x^2 - 2x - 1$ ***(M2)***

Therefore, to degree 3, $\dfrac{2x^3}{3} - x^2 - 2x - 1$ ***(A1)***

[3 marks]

(c) Points of intersection: $(x^2 - 1)\mathrm{e}^{kx} = \mathrm{e}^{kx} \Rightarrow \mathrm{e}^{kx}(x^2 - 2) = 0 \Rightarrow x = \pm\sqrt{2}$ ***(M1)(A1)***

Area: $\int_{-\sqrt{2}}^{\sqrt{2}} \mathrm{e}^{kx}\left(2 - x^2\right)\mathrm{d}x$ ***(C1)***

$= \dfrac{\sqrt{2} + \sqrt{2}}{8}\left[0 + 0 + 2\left(\tfrac{3}{2}\mathrm{e}^{-\frac{k\sqrt{2}}{2}} + 2 + \tfrac{3}{2}\mathrm{e}^{\frac{k\sqrt{2}}{2}}\right)\right]$ ***(M1)(A1)***

$= \dfrac{\sqrt{2}}{4}\left[3\mathrm{e}^{-\frac{k\sqrt{2}}{2}} + 3\mathrm{e}^{\frac{k\sqrt{2}}{2}} + 4\right]$ ***(C1)***

[6 marks]

[Total: 30 marks]

10. (i) (a) The arcs corresponding to [ED], [DC], [CB], and [BA] are half of the whole circle, ***(M1)***
$\Rightarrow$ ECA is a semicircle. $\Rightarrow \mathrm{E\hat{O}A} = 180°, \Rightarrow$ E, O, and A are collinear. ***(R1)(AG)***

[2 marks]

(b) [EA] is a diameter, $\Rightarrow \mathrm{E\hat{B}A} = 90°$ and $(\mathrm{OK}) \perp (\mathrm{AB})$ ***(R1)***
$\Rightarrow$ (OK) parallel to (EB) ***(C1)***
$\Rightarrow$ OK = ½EB (line through midpoint of a side parallel to another side.) ***(R1)***
Similarly, OJ = ½AD ***(C1)***

[4 marks]

(c) [DB] is common to both triangles. ***(R1)***
Since DC = DE, the arcs corresponding to them are equal. ***(R1)***
Hence angles CBD and EBD are equal. ***(R1)***
$\Delta\mathrm{DCB} \cong \Delta\mathrm{DIB}$ by ASA ***(R1)***

[4 marks]

(d) In ΔIBA: $\mathrm{I\hat{B}A} = 90°$, CB=IB $\Rightarrow$ IB=3 and BA=3. ***(M1)(C1)***
$\Rightarrow \Delta$IBA is an isosceles right-angled triangle.
Similar arguments for ΔDIE. ***(C1)***

[3 marks]

(e) Using Pythagoras' theorem, $\mathrm{AI} = 3\sqrt{2}$, and $\mathrm{EI} = 2\sqrt{2}$ ***(C1)(C1)***

[2 marks]

(f) Since $\mathrm{EB} = \mathrm{EI} + \mathrm{IB} = 3 + 2\sqrt{2}$ ***(M1)***

$\Rightarrow \mathrm{OK} = \tfrac{1}{2}\left(3 + 2\sqrt{2}\right)$ ***(A1)***

$\Rightarrow \Delta\mathrm{OAB} = \tfrac{1}{2} \cdot 3 \cdot \tfrac{1}{2} \cdot \left(3 + 2\sqrt{2}\right) = \tfrac{3}{4}\left(3 + 2\sqrt{2}\right)$

Also $\mathrm{OJ} = \tfrac{1}{2}\mathrm{AD} = \tfrac{1}{2}(\mathrm{IA} + \mathrm{ID}) = \tfrac{1}{2}\left(2 + 3\sqrt{2}\right)$ ***(A1)***

$\Rightarrow \Delta\mathrm{OED} = \tfrac{1}{2} \cdot 2 \cdot \tfrac{1}{2}\left(2 + 3\sqrt{2}\right) = \tfrac{2}{4}\left(2 + 3\sqrt{2}\right)$

Area of the octagon $= 4\left[\tfrac{3}{4}\left(3 + 2\sqrt{2}\right) + \tfrac{2}{4}\left(2 + 3\sqrt{2}\right)\right]$ ***(M1)***

$= 13 + 12\sqrt{2}$ ***(A1)***

[5 marks]

continued...

Question 10 continued

(ii) (a) Let M(*x, y*) be any point on the circle.

$\vec{\mathrm{MP}} \perp \vec{\mathrm{MQ}} \Rightarrow \vec{\mathrm{MP}}.\vec{\mathrm{MQ}} = 0$ ***(M1)***

Since $\Rightarrow (x-1)(x+3)+(y-3)(y-9)=0$ ***(M1)***

$\Rightarrow x^2+2x+y^2-12y+24=0$ ***(A1)***

[3 marks]

(b) $\mathrm{MP}^2 = 9\mathrm{MQ}^2 \Rightarrow (x+3)^2+(y-9)^2 = 9\left[(x-1)^2+(y-3)^2\right]$ ***(M1)***

$\Rightarrow 8x^2-24x+8y^2-36y=0$ ***(M1)***

$\Rightarrow \left(x-\frac{3}{2}\right)^2+\left(y-\frac{9}{4}\right)^2=\frac{117}{16}$ ***(A1)***

This a circle with centre $\left(\frac{3}{2},\frac{9}{4}\right)$, and radius $\frac{\sqrt{117}}{4}$ ***(C1)***

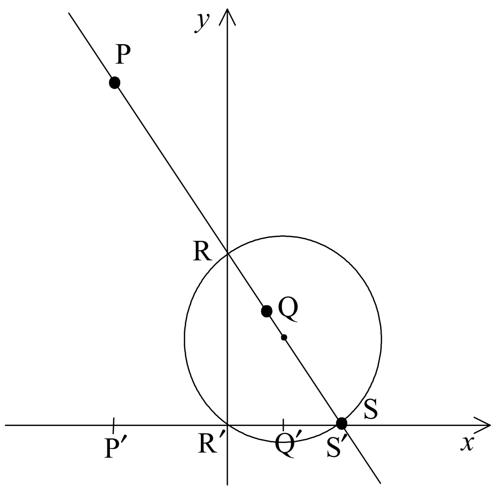
[4 marks]

(c) Equation of PQ: $y=-\frac{3}{2}x+\frac{9}{2}$, therefore the point of intersection must also satisfy the equation of the circle, hence:

$\left(x-\frac{3}{2}\right)^2+\left(-\frac{3}{2}x+\frac{9}{2}-\frac{9}{4}\right)^2=\frac{117}{16}\Rightarrow x^2-3x=0$ ***(M1)(A1)***

$\Rightarrow x=0$, or $x=3$

Let $\mathrm{P}', \mathrm{Q}', \mathrm{R}', \mathrm{S}'$ be the projections of P, Q, R, S on to the *x*-axis $(\mathrm{S}=\mathrm{S}')$



$\frac{\mathrm{R'P'}}{\mathrm{R'Q'}}=-\frac{3}{1};\ \frac{\mathrm{S'P'}}{\mathrm{S'Q'}}=\frac{6}{2}\Rightarrow\frac{\mathrm{R'P'}}{\mathrm{R'Q'}}=-\frac{\mathrm{S'P'}}{\mathrm{S'Q'}}\ (=-3)$ ***(M1)***

$\Rightarrow \mathrm{P}', \mathrm{Q}', \mathrm{R}',$ and S' are in a harmonic ratio

Therefore, P, Q, R, and S are in a harmonic ratio. ***(AG)***

[3 marks]

[Total: 30 marks]