N99/510/H(2)

INTERNATIONAL BACCALAUREATE
BACCALAURÉAT INTERNATIONAL
BACHILLERATO INTERNACIONAL

MATHEMATICS

Higher Level

Friday 5 November 1999 (morning)

Paper 2 — 2 hours 30 minutes

This examination paper consists of 2 sections, Section A and Section B.
Section A consists of 4 questions.
Section B consists of 4 questions.
The maximum mark for Section A is 80.
The maximum mark for each question in Section B is 40.
The maximum mark for this paper is 120.

INSTRUCTIONS TO CANDIDATES

Do NOT open this examination paper until instructed to do so.

Answer all FOUR questions from Section A and ONE question from Section B.

Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures as appropriate.

EXAMINATION MATERIALS

Required:
IB Statistical Tables
Millimetre square graph paper
Calculator
Ruler and compasses

Allowed:
A simple translating dictionary for candidates not working in their own language

FORMULAE

Trigonometrical identities:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = 2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$

$$\text{If } \tan\frac{\theta}{2} = t \text{ then } \sin\theta = \frac{2t}{1+t^2} \text{ and } \cos\theta = \frac{1-t^2}{1+t^2}$$

Integration by parts:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Standard integrals:

$$\int \frac{\mathrm{d}x}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + c$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + c \quad (|x| < a)$$

Statistics: If $(x_1, x_2, \ldots, x_n)$ occur with frequencies $(f_1, f_2, \ldots, f_n)$ then the mean m and standard deviation s are given by

$$m = \frac{\Sigma f_i x_i}{\Sigma f_i}, \qquad s = \sqrt{\frac{\Sigma f_i (x_i - m)^2}{\Sigma f_i}}, \qquad i = 1, 2, \ldots, n$$

Binomial distribution:

$$p_x = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

A correct answer with **no** *indication of the method used will normally receive* **no** *marks. You are therefore advised to show your working.*

SECTION A

Answer all FOUR questions from this section.

1. *[Maximum mark: 22]*

(i) (a) Evaluate $(1 + \mathrm{i})^2$, where $\mathrm{i} = \sqrt{-1}$. *[2 marks]*

(b) Prove, by mathematical induction, that $(1 + \mathrm{i})^{4n} = (-4)^n$, where $n \in \mathbb{N}^*$. *[6 marks]*

(c) Hence or otherwise, find $(1 + \mathrm{i})^{32}$. *[2 marks]*

(ii) Let $z_1 = \dfrac{\sqrt{6} - \mathrm{i}\sqrt{2}}{2}$, and $z_2 = 1 - \mathrm{i}$.

(a) Write z_1 and z_2 in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2}$. *[6 marks]*

(b) Show that $\dfrac{z_1}{z_2} = \cos\dfrac{\pi}{12} + \mathrm{i}\sin\dfrac{\pi}{12}$. *[2 marks]*

(c) Find the value of $\dfrac{z_1}{z_2}$ in the form $a + b\mathrm{i}$, where a and b are to be determined exactly in radical (surd) form. Hence or otherwise find the exact values of $\cos\dfrac{\pi}{12}$ and $\sin\dfrac{\pi}{12}$. *[4 marks]*

2. *[Maximum mark: 21]*

Consider the points A(1 , 2 , 1) , B(0 , –1 , 2) , C(1 , 0 , 2) , and D(2 , –1 , –6) .

(a) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$. *[2 marks]*

(b) Calculate $\overrightarrow{AB} \times \overrightarrow{BC}$. *[3 marks]*

(c) Hence, or otherwise find the area of triangle ABC . *[2 marks]*

(d) Find the equation of the plane P containing the points A , B , and C . *[3 marks]*

(e) Find a set of parametric equations for the line through the point D and perpendicular to the plane P. *[2 marks]*

(f) Find the distance from the point D to the plane P . *[3 marks]*

(g) Find a unit vector which is perpendicular to the plane P . *[2 marks]*

(h) The point E is a reflection of D in the plane P. Find the coordinates of E . *[4 marks]*

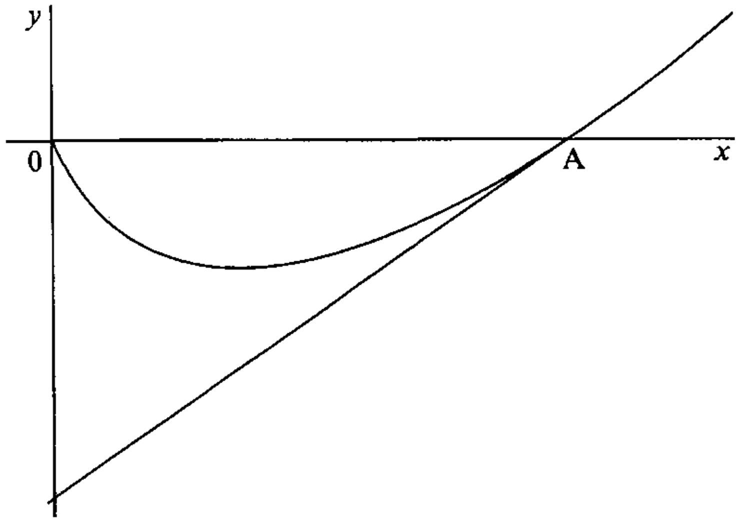
3. *[Maximum mark: 20]*

Consider the function $f_k(x) = \begin{cases} x \ln x - kx, & x > 0 \\ 0, & x = 0 \end{cases}$, where $k \in \mathbb{N}$

(a) Find the derivative of $f_k(x)$, $x > 0$. *[2 marks]*

(b) Find the interval over which $f_0(x)$ is increasing. *[2 marks]*

The graph of the function $f_k(x)$ is shown below.



(c) (i) Show that the stationary point of $f_k(x)$ is at $x = e^{k-1}$.

(ii) One x-intercept is at $(0, 0)$. Find the coordinates of the other x-intercept. *[4 marks]*

(d) Find the area enclosed by the curve and the x-axis. *[5 marks]*

(e) Find the equation of the tangent to the curve at A. *[2 marks]*

(f) Show that the area of the triangular region created by the tangent and the coordinate axes is twice the area enclosed by the curve and the x-axis. *[2 marks]*

(g) Show that the x-intercepts of $f_k(x)$ for consecutive values of k form a geometric sequence. *[3 marks]*

4. *[Maximum marks: 17]*

The continuous random variable X has probability density function $f(x)$ where

$$f(x) = \begin{cases} e - ke^{kx}, & 0 \leqslant x \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that $k = 1$. *[3 marks]*

(b) What is the probability that the random variable X has a value that lies between $\frac{1}{4}$ and $\frac{1}{2}$? Give your answer in terms of e. *[2 marks]*

(c) Find the mean and variance of the distribution. Give your answers *exactly*, in terms of e. *[6 marks]*

The random variable X above represents the lifetime, in years, of a certain type of battery.

(d) Find the probability that a battery lasts more than six months. *[2 marks]*

A calculator is fitted with three of these batteries. Each battery fails independently of the other two. Find the probability that at the end of six months

(e) none of the batteries has failed; *[2 marks]*

(f) exactly one of the batteries has failed. *[2 marks]*

SECTION B

Answer ONE question from this section.

Abstract Algebra

5. *[Maximum mark: 40]*

(i) Consider the set $U = \{1, 3, 5, 9, 11, 13\}$ under the operation $*$, where $*$ is multiplication modulo 14. (In all parts of this problem, the general properties of multiplication modulo n may be assumed.)

(a) Show that $(3 * 9) * 13 = 3 * (9 * 13)$. *[2 marks]*

(b) Show that $(U, *)$ is a group. *[11 marks]*

(c) (i) Define a cyclic group. *[2 marks]*

(ii) Show that $(U, *)$ is cyclic and find all its generators. *[7 marks]*

(d) Show that there are only two non-trivial proper subgroups of this group, and find them. *[7 marks]*

(ii) Consider a group $(G, \circ)$ with identity e. Suppose that H is a subset of G such that $H = \{x \in G : x \circ a = a \circ x \text{, for all } a \in G\}$.

Show that $(H, \circ)$ is a subgroup of $(G, \circ)$, by showing that

(a) $e \in H$; *[2 marks]*

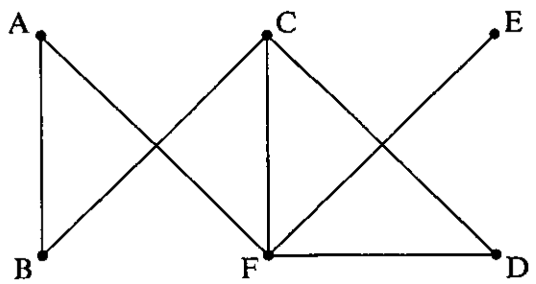
(b) if $x, y \in H$, then $x \circ y \in H$,
[i.e. show that $(x \circ y) \circ a = a \circ (x \circ y)$]; *[5 marks]*

(c) if $x \in H$, then $x^{-1} \in H$. *[4 marks]*

Graphs and Trees

6. *[Maximum mark: 40]*

(i) The network of cities in a certain region served by an airline is shown in the following diagram, with edges representing direct connections.



(a) Copy and complete the table below which shows the least number of edges connecting each pair of cities in this network. (Such a table gives the least number of stops required between cities in this region.) *[4 marks]*

	A	B	C	D	E	F
A						
B			1			
C						
D						
E				2		
F						

(b) The *accessibility index* is used to determine how easy it is to get to a particular city. To calculate the accessibility index for a given city, the total of each column is divided by the degree of the vertex representing it. The most accessible city is the one with the smallest accessibility index, and the least accessible city has the largest accessibility index.

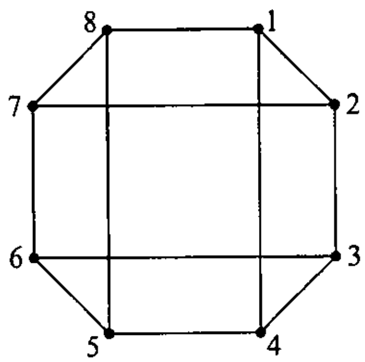
(i) Which city in this region is the most accessible and which city is the least accessible? Give your reasons. *[4 marks]*

(ii) A new flight is added between cities A and C. With this change, which city is the most accessible and which city is the least accessible? Show your working. *[3 marks]*

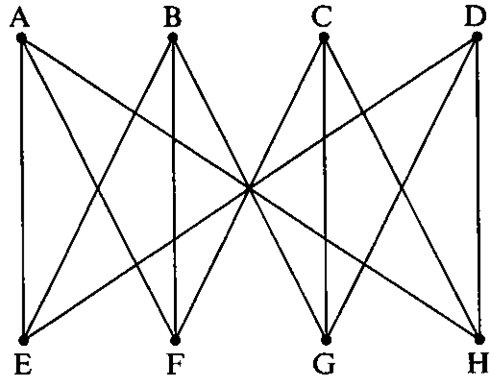
(This question continues on the following page)

(Question 6 continued)

(ii) Below are two graphs U and V with 8 vertices each.



U



V

(a) Set up an adjacency matrix for graph U . *[3 marks]*

(b) By setting up an appropriate adjacency matrix for V , show that the two graphs are isomorphic. *[6 marks]*

(c) Determine whether V is planar. *[4 marks]*

(This question continues on the following page)

(*Question 6 continued*)

(iii) *In this part, marks will only be awarded if you show the correct application of the required algorithms, and show all your working.*

In an offshore drilling site for a large oil company, the distances between the planned wells are given below in metres.

	1	2	3	4	5	6	7	8	9	10
2	30									
3	40	60								
4	90	190	130							
5	80	200	10	160						
6	70	40	20	40	130					
7	60	120	50	90	30	60				
8	50	140	90	70	140	70	40			
9	40	170	140	60	50	90	50	70		
10	200	80	150	110	90	30	190	90	100	
11	150	30	200	120	190	120	60	190	150	200

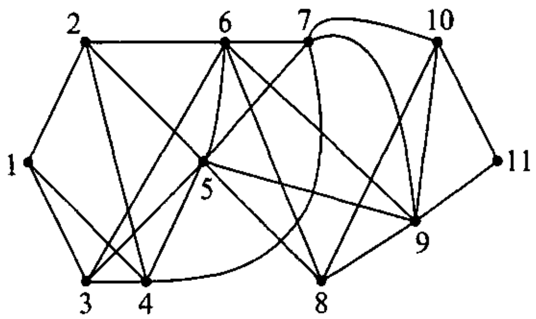
(a) It is intended to construct a network of paths to connect the different wells in a way that minimises the sum of the distances between them.

Use Prim's algorithm to find a network of paths of minimum total length that can span the whole site. *[8 marks]*

(This question continues on the following page)

(Question 6 continued)

(b) Pipes are laid under water. Well 1 has the largest amount of oil to be pumped per day, and Well 11 is designed to be the main transportation hub. The only possible connections to be made between wells are shown in the diagram below.



The associated cost for each pipe, in 100-thousand dollar figures, are given in the table below. Use Dijkstra's algorithm to find the path with minimum cost that can transport oil from Well 1 to Well 11 . *[8 marks]*

	1	2	3	4	5	6	7	8	9	10
2	6									
3	3									
4	8	7	2							
5		14	12	6						
6		16	19		7					
7				24	20	29				
8					23	15				
9					56	30	41	50		
10							42	25	40	
11									32	22

Statistics

7. *[Maximum mark: 40]*

(i) An automated coffee machine is set to dispense coffee into cups. The amount of coffee dispensed by the machine into each cup can be regarded as being normally distributed, with a mean of 330 ml and standard deviation of 6 ml .

(a) What is the probability that the amount of coffee dispensed into a cup exceeds 327 ml ? *[2 marks]*

(b) What amount is exceeded by 95 % of the cups? *[3 marks]*

The vendor of this machine suspects that the machine is dispensing more than the desired amount. For that purpose, she collects data about 60 cups and finds that the average amount per cup was 332 ml .

(c) What is the probability that this machine with its current settings can give a sample with average 332 ml or higher? *[3 marks]*

(d) Hence or otherwise, test at the 5 % level of significance, the vendor's hypothesis that the machine is dispensing more than the desired amount of 330 ml . *[3 marks]*

Still not convinced of the results, the vendor asks one of her managers to measure the amount of coffee in a random sample of 25 other cups. The results were as follows:

334.993	342.237	327.654	336.643	326.982
342.208	329.612	336.626	338.207	329.834
330.679	331.704	330.974	334.564	324.996
331.612	341.288	318.524	333.665	335.889
324.537	337.487	335.379	330.758	332.540

(e) Given that this sample has a mean of 332.78 ml (correct to 2 decimal places) , find a 95 % confidence interval for the true population mean. Give your answer correct to 2 decimal places. *[4 marks]*

(f) Hence or otherwise test, at the 5 % level of significance, the hypothesis that the machine is not dispensing the required amount. *[2 marks]*

(g) Another objective of this second exercise is to determine whether the proportion of cups that are overfilled is more than 0.5. Find a 95 % confidence interval for the true proportion of cups that have more coffee than the vendor desires. *[4 marks]*

(*This question continues on the following page*)

(Question 7 continued)

(ii) In an effort to study the level of intelligence of students entering college, a psychologist collected data from 4000 students who were given a standard test. The predictive norms for this particular test were computed from a very large population of scores having a normal distribution with mean 100 and standard deviation of 10. The psychologist wishes to determine whether the 4000 test scores he obtained also came from a normal distribution with mean 100 and standard deviation 10. He prepared the following table (Expected frequencies are rounded to the nearest integer):

Score	observed frequencies	expected frequencies	Score	observed frequencies	expected frequencies
$\leqslant 70.5$	20	6	100.5–110.5	1450	
70.5– 80.5	90	96	110.5–120.5	499	507
80.5– 90.5	575		120.5–130.5	80	76
90.5–100.5	1282		$\geqslant 130.5$	4	4

(a) Copy and complete the table, showing how you arrived at your answers. *[5 marks]*

(b) Test the hypothesis at the 5 % level of significance. *[6 marks]*

(iii) The International Medical Association claims that the proportion of people who do not suffer from colds during the winter months is higher for those who received flu immunisation injections than those who do not have flu immunisation injections. The following table gives data collected from a random sample of 100 people.

	No flu immunisation injections	Flu immunisation injections
Colds	35	5
No colds	45	15

Does the claim appear to be justified at the 5 % level of significance? *[8 marks]*

Analysis and Approximation

8. *[Maximum mark: 40]*

(i) Consider the function $f(x) = 2x^3 - 15x^2 + 36x - 26$.

(a) Evaluate $f(1)$ and $f(2)$. *[2 marks]*

(b) Find $f'(x)$ *[1 mark]*

(c) Show that $f(x)$ has **exactly** one zero for $1 < x < 2$. *[3 marks]*

(d) Use the Newton-Raphson method to estimate this zero, giving your answer correct to 3 decimal places. Take $x_0 = 1.5$ as a first estimate. *[4 marks]*

(e) Write the equation $2x^3 - 15x^2 + 36x - 26 = 0$ in the form $x = g(x) = \sqrt[3]{h(x)}$, where $h(x)$ is to be determined. Then use three iterations of $x_{n+1} = g(x_n)$ with $x_0 = 1.5$ to approximate the zero of this function. *[4 marks]*

(ii) Discuss the convergence or divergence of the following series:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ *[4 marks]*

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + k}$, k is a positive integer. *[5 marks]*

(This question continues on the following page)

(*Question 8 continued*)

(iii) (a) Consider the function $f(x) = \sqrt{2 - x^2}$. Show that

$$f''(x) = \frac{2}{(x^2 - 2)\sqrt{2 - x^2}} .$$

[3 marks]

(b) The third and fourth derivatives of $f(x)$ with respect to x are given by:

$$f'''(x) = \frac{-6x}{(x^2 - 2)^2\sqrt{2 - x^2}} \text{ and } f^{(4)}(x) = \frac{12(2x^2 + 1)}{(x^2 - 2)^3\sqrt{2 - x^2}} .$$

Use **Maclaurin's series** to expand $f(x)$, and to show that

$\sqrt{2 - x^2} \approx \sqrt{2}\left(1 - \frac{x^2}{4} + kx^4\right)$. Find the value of k. *[5 marks]*

(c) Hence or otherwise show that the value of $\int_0^1 x^2\sqrt{2 - x^2}\,dx$ is approximately 0.39438 . *[4 marks]*

(d) Use the substitution $x = \sqrt{2}\sin\theta$ to show that the exact value of $\int_0^1 x^2\sqrt{2 - x^2}\,dx$ is $\frac{\pi}{8}$. Hence find an approximation of π to 4 decimal places. *[5 marks]*