INTERNATIONAL BACCALAUREATE

MARKSCHEME

May 1997

MATHEMATICS

Higher Level

Paper 1

1. $$AB = \begin{pmatrix} 1 & p & q \\ 1 & 0 & r \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2+3p & p & q \\ 2 & 0 & r \\ 0 & 0 & 2 \end{pmatrix}$$ *(M1)(A1)*

Hence $AB = C$ implies

$$\begin{pmatrix} 2+3p & p & q \\ 2 & 0 & r \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 10 \\ 2 & 0 & -6 \\ 0 & 0 & 2 \end{pmatrix}$$

So $2+3p=-1, p=-1, q=10$ and $r=-6$ *(R1)*

$p=-1, q=10, r=-6$ *(A1)*

Answers: $p=-1, q=10, r=-6$ *[C4]*

2. (a) $\vec{OP} = -i+5j+7k,\ \vec{OQ} = i+2j+3k$ *(C1)*

(b) $\vec{OP}\times\vec{OQ} = \begin{vmatrix} i & j & k \\ -1 & 5 & 7 \\ 1 & 2 & 3 \end{vmatrix} = i+10j-7k$ *(C2)*

(c) Area of the parallelogram is $|\vec{OP}\times\vec{OQ}|$

$= |i+10j-7k| = \sqrt{150} = 5\sqrt{6}$ *(C1)*

Answers: (a) $\vec{OP} = -i+5j+7k, \vec{OQ} = i+2j+3k$

(b) $\vec{OP}\times\vec{OQ} = i+10j-7k$

(c) Area of the parallelogram $= \sqrt{150} = 5\sqrt{6}$ *[C4]*

Note: In (a) award ***(C1)*** only if both $\vec{OP}$ and $\vec{OQ}$ are correct, otherwise award ***(C0)***.

3. $z = -1 + 3\mathrm{i},\ w = 3 + \mathrm{i}$

$zw = (-1+3\mathrm{i})(3+\mathrm{i}) = -6 + 8\mathrm{i}$ *(M1)(A1)*

$\frac{z}{w} = \frac{(-1+3\mathrm{i})(3-\mathrm{i})}{(3+\mathrm{i})(3-\mathrm{i})} = \frac{(-3+3)+(9+1)\mathrm{i}}{10} = \frac{0+10\mathrm{i}}{10} = 0 + \mathrm{i} = \mathrm{i}$ *(M1)(A1)*

Answers: $zw = -6 + 8\mathrm{i}$ *[C4]*

$\frac{z}{w} = 0 + \mathrm{i}$ or i

4. Probability of drawing a black and a white ball is $= \frac{\binom{5}{1}\binom{7}{1}}{\binom{12}{2}} = \frac{(5)(7)}{66} = \frac{35}{66}$ *(M2)(A2)*

Answer: $\frac{35}{66}$ *[C4]*

Remark: Some candidates may solve the problem as follows:
Probability of choosing a white ball first and a black ball next is

$\left(\frac{5}{12}\right)\left(\frac{7}{11}\right) = \frac{35}{132}$. *(M1)(A1)*

Probability of choosing a black ball first and next a white ball is

$\left(\frac{7}{12}\right)\left(\frac{5}{11}\right) = \frac{35}{132}$. *(M1)*

Hence probability of choosing a black and a white ball in two draws is

$\frac{35}{122} + \frac{35}{122} = \frac{35}{66}$. *(A1)*

5. Area $= \int_0^k \sqrt{x}\,\mathrm{d}x = \left[\frac{x^{3/2}}{3/2}\right]_0^k = \frac{2}{3}k^{3/2}$ *(M2)(A2)*

Answer: Area $= \frac{2}{3}k^{3/2}$ *[C4]*

6. $f'(x) = \ln x + \frac{x}{x} + e^{\sin x}\cos x + \frac{1}{1+x^2}$ (C1)(C1)(C1)(C1)

$= \ln x + 1 + e^{\sin x}\cos x + \frac{1}{1+x^2}$

Answer: $f'(x) = 1 + \ln x + e^{\sin x}\cos x + \frac{1}{1+x^2}$ *[C4]*

7. $\int (x^2-1)^3 x\,dx = \frac{1}{2}\int (x^2-1)^3 2x\,dx = \frac{(x^2-1)^4}{8} + c$ (M2)(A2)

Answer: $\frac{(x^2-1)^4}{8} + c$ *[C4]*

8. Probability of a student not graduating = 0.3
Probability of a student graduating $= 1 - 0.3 = 0.7$ *(M1)*
Probability that exactly 4 out of 6 of the randomly selected students graduate is
$\binom{6}{4}(0.7)^4(0.3)^2 \approx 0.324$. *(M1)(A2)*

Answer: 0.324 *[C4]*

9. $3\sin^2\theta - 7\sin\theta + 5 = 3 - 3\sin^2\theta \quad (0^\circ \le \theta \le 90^\circ)$

or: $6\sin^2\theta - 7\sin\theta + 2 = 0$
or: $(3\sin\theta - 2)(2\sin\theta - 1) = 0$
or: $\sin\theta = \frac{2}{3}$ or $\sin\theta = \frac{1}{2}$ *(M1)(A1)*

Hence, $\theta = \arcsin\frac{2}{3} \approx 41.8^\circ$

or, $\theta = \arcsin\frac{1}{2} = 30^\circ$ *(M1)(A1)*

Answer: $\theta = 41.8^\circ$ or 30° *[C4]*

10. $z = 1 + 2i$ is a solution implies $z = 1 - 2i$ is also a solution. Hence $[z-(1+2i)][z-(1-2i)] = z^2 - 2z + 5$ is a factor of $z^3 - 3z^2 + 7z - 5 = 0$. *(M1)(A1)*

The other factor is $\dfrac{z^3 - 3z^2 + 7z - 5}{z^2 - 2z + 5} = z - 1$ *(M1)*

Hence the other two solutions are $z = 1 - 2i$ and $z = 1$. *(A1)*

Answers: $z = 1 - 2i,\ z = 1$ *[C4]*

11. $\vec{OP} = 3i - j$ and $\vec{OQ} = \lambda i - (\lambda + 4)j$ *(A1)(A1)*

$\vec{OP} \cdot \vec{OQ} = 4\lambda + 4$

Since $\vec{OP}$ is perpendicular to $\vec{OQ}$, $4\lambda + 4 = 0$

Hence $\lambda = -1$ *(M1)(A1)*

Answer: $\lambda = -1$ *[C4]*

12. Probability that a component is produced by machine $A = p(A) = \dfrac{2500}{4000} = \dfrac{5}{8}$.

Probability that a component is produced by machine $B = p(B) = \dfrac{1500}{4000} = \dfrac{3}{8}$.

Probability that a component is faulty is $p(F)$.

(a) $p(F) = p(A)p(F \mid A) + p(B)p(F \mid B) = \dfrac{5}{8}(0.04) + \dfrac{3}{8}(0.05) = \dfrac{7}{160}$ *(M1)(A1)*

$= 0.04375$

(b) $p(A \mid F) = \dfrac{p(A)p(F \mid A)}{p(F)} = \dfrac{\left(\dfrac{5}{8}\right)(0.04)}{0.04375} = \dfrac{4}{7} \approx 0.571$ *(M1)(A1)*

Answers: (a) 0.04375 *[C2]*
(b) 0.571 *[C2]*

13. $\int x^2 e^{-2x} dx = -\frac{1}{2}e^{-2x}x^2 + \frac{1}{2}\int e^{-2x} 2x dx$

$$= -\frac{1}{2}e^{-2x}x^2 + \int e^{-2x} x dx \qquad \textit{(C1)(C1)}$$

$$= -\frac{1}{2}e^{-2x}x^2 - \frac{1}{2}e^{-2x}x + \frac{1}{2}\int e^{-2x} dx \qquad \textit{(C1)}$$

$$= -\frac{1}{2}e^{-2x}\left(x^2 + x + \frac{1}{2}\right) + c \qquad \textit{(C1)}$$

Answer: $-\frac{1}{2}e^{-2x}\left(x^2 + x + \frac{1}{2}\right) + c$ *[C4]*

14. We want to find L such that $p\left(z \geq \frac{0.70 - L}{0.12}\right) = 0.2$.

The closest probability to 0.8 from the table of standard normal probabilities is 0.7995 and the corresponding z value is 0.84. *(R1)(A1)*

Hence, $\frac{0.70 - L}{0.12} = 0.84$

and $L = 0.70 - (0.84)(0.12) \approx 0.599$ *(M1)(A1)*

Answer: $L = 0.599$ *[C4]*

15. Let the kth term in the binomial expansion of $\left(\frac{9}{2}x^2 - \frac{1}{9x}\right)^9$ be independent of x.

Then $\binom{9}{k}\left(\frac{9}{2}x^2\right)^k\left(-\frac{1}{9x}\right)^{9-k}$ is independent of x.

Thus $\left(x^{2k}\right)\left(x^{k-9}\right) = x^0$ or $3k - 9 = 0$ or $k = 3$. *(M1)(A1)*

Hence, the coefficient of the term independent of x is $\binom{9}{3}\left(\frac{9}{2}\right)^3\left(-\frac{1}{9}\right)^6 = \frac{7}{486}$. *(M1)(A1)*

Answer: $\frac{7}{486}$ *[C4]*

16. Differentiate $xy^2 + x^2y = 2$, implicitly with respect to x.

Hence $y^2 + 2xyy' + 2xy + x^2y' = 0$

or: $y' = -(2xy + y^2)/(2xy + x^2)$ *(M1)*

(a) Thus, the gradient of the curve at (1, 1) is -1. *(A1)*

(b) Slope of the line perpendicular to the centre at (1, 1) is 1 *(C1)*

Equation of the line perpendicular to the curve at (1, 1) is $y - 1 = 1(x - 1)$ or $y = x$ *(C1)*

Answers: (a) Gradient $= -1$ *[C2]*

(b) Equation of the line is $y = x$ *[C2]*

17. (a) Integrating factor is $e^{\tan x}$. *(C1)*

(b) Multiplying the given equation by the integrating factor, we get,

$y'e^{\tan x} + ye^{\tan x}\sec^2 x = e^{\tan x}\sec^2 x$

or: $\frac{d}{dx}(ye^{\tan x}) = e^{\tan x}\sec^2 x$

or: $ye^{\tan x} = \int e^{\tan x}\sec^2 x dx = e^{\tan x} + k$

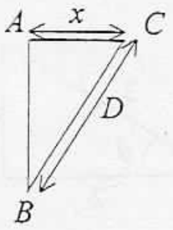
or: $y = 1 + ke^{-\tan x}$ *(M1)(A1)*

Since $y = 2$ when $x = 0$, we have

$2 = 1 + ke^0$ or $k = 1$. *(R1)*

The solution over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is $y = 1 + e^{-\tan x}$.

18. Let B be the radar station, A the position of the airplane when it was vertically above the radar station and C be the position of the airplane after an elapse of t hours.



Given $AB = 10\,000$ metres $= 10$ km. Let $AC = x$ km. Distance, D, of the airplane from the radar station after t hours $= BC = \sqrt{10^2 + x^2}$.

$$\frac{dD}{dt} = \frac{x}{\sqrt{100+x^2}}\frac{dx}{dt} = \left(\frac{x}{\sqrt{100+x^2}}\right)800$$ *(M1)(A1)*

Hence, when $x = 2$ km,

$$\frac{dD}{dt} = \left(\frac{2}{\sqrt{104}}\right)800 = \frac{800}{\sqrt{26}} \text{ km / hour}$$ *(M1)(A1)*

Answer: $\frac{800}{\sqrt{26}}$ km / hour *[C4]*

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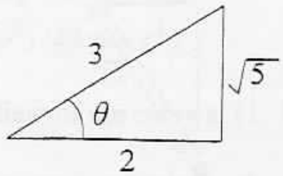
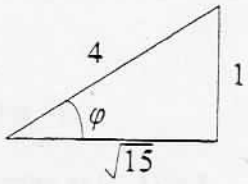
19. Let $\theta = \arccos\frac{2}{3}$ and $\varphi = \arcsin\frac{1}{4}$. Then





$$\sin(\arccos\frac{2}{3} + \arcsin\frac{1}{4}) = \sin(\theta + \varphi)$$ *(M1)*

$$= \sin\theta\cos\varphi + \cos\theta\sin\varphi = \left(\frac{\sqrt{5}}{3}\right)\left(\frac{\sqrt{15}}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{2}{3}\right)$$ *(M1)(A1)*

$$= \frac{\sqrt{75}+2}{12} = \frac{5\sqrt{3}+2}{12}$$ *(A1)*

Answer: $\frac{5\sqrt{3}+2}{12}$ *[C4]*

20. (a) Period =3 *(C1)*

(b) Since $52 = 1(\bmod 3)$, $f(52) = f(1) = 1$ *(M1)(A1)*

(c) $f'(26\frac{1}{2}) = f'(24 + 2\frac{1}{2}) = f'(2\frac{1}{2}) = -2$ *(C1)*

Answers:
(a) 3
(b) 1
(c) –2 *[C4]*