1. (i) (a) Line L meets plane P when $2t + 3 + 3(4t - 1) - 2(2 - t) = 4$

$\Rightarrow t = \frac{1}{2}$.

The required point of intersection is $\left(4, 1, 1\frac{1}{2}\right)$. *(M1) (A1)*

(b) $\vec{a} = 2\vec{i} + 4\vec{j} - \vec{k}$ is parallel to line L, and $\vec{n} = \vec{i} + 3\vec{j} - 2\vec{k}$ is normal to plane P.

If θ is the required angle, then

$$\sin\theta = \frac{|\vec{a} \cdot \vec{n}|}{|\vec{a}|\,|\vec{n}|}$$

$$= \frac{16}{\sqrt{21}\,\sqrt{14}}$$

$$= \frac{16}{7\sqrt{6}}$$

The required angle = 68.9° (nearest 0.1°). *(M2) (A1)*

(c) $\vec{b} = -4\vec{i} + 2\vec{j} + \vec{k}$ is parallel to line M, and $\vec{b} \cdot \vec{n} = 0$.

Therefore, **line M is parallel to plane P.** *(R1) (A1)*

(d) If line L meets line M, the equations

$$2t + 3 = 6 - 4u$$
$$4t - 1 = 2u$$
$$2 - t = u - 10$$

must have a unique solution.

From the first and second, $t = \frac{1}{2}$, $u = \frac{1}{2}$.

Now, these do **not** satisfy the third equation $\left(2 - \frac{1}{2} \neq \frac{1}{2} - 10\right)$.

Therefore, **lines L and M do not meet.** *(M2) (A2)*

(e) **Method 1**

Let $A(2t + 3,\ 4t - 1,\ 2 - t)$ lie on L and
$B(6 - 4u,\ 2u,\ u - 10)$ lie on M.

Now, $\vec{AB} = \begin{pmatrix} 3 - 4u - 2t \\ 1 + 2u - 4t \\ -12 + u + t \end{pmatrix}$. *(M1) (A1)*

Also, the vector $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 4 & -1 \\ -4 & 2 & 1 \end{vmatrix} = 6\vec{i} + 2\vec{j} + 20\vec{k}$ is perpendicular to both lines L and M. *(A1)*

If A and B are the points which are closest together, $\vec{AB}$ would be parallel to $\vec{a} \times \vec{b}$. *(R1)*

Therefore, $3 - 4u - 2t = 3(1 + 2u - 4t) \Rightarrow 10t - 10u = 0$

and $-12 + u + t = 10(1 + 2u - 4t) \Rightarrow 41t - 19u = 22$.

Solving gives $u = t = 1$, and so $\vec{AB} = \begin{pmatrix} -3 \\ -1 \\ -10 \end{pmatrix}$.

Therefore, **the shortest distance** $= \sqrt{110}$**.** *(A1)*

Method 2

$\vec{a} \times \vec{b} = 6\vec{i} + 2\vec{j} + 20\vec{k}$ is perpendicular to both lines L and M. *(A1)*

The plane $3x + y + 10z = 28$ contains the line L and is parallel to M. *(R1)*

Now, the required distance is the distance from the point $(6, 0, -10)$, which lies on M, to this plane.

The required distance $= \dfrac{|18 + 0 - 100 - 28|}{\sqrt{110}} = \sqrt{110}$, *(M2) (A1)*

Method 3

$\vec{\omega} = 3\vec{i} + \vec{j} + 10\vec{k}$ is perpendicular to both lines. *(A1)*

$A\,(3, -1, 2)$ lies on L; $B\,(6, 0, -10)$ lies on M. *(A1)*

The required distance is the orthogonal projection of $\vec{AB}$ on $\vec{\omega}$. ***(R1)***

This is $\dfrac{|\vec{AB}\cdot\vec{\omega}|}{|\vec{\omega}|} = \dfrac{1}{\sqrt{110}}\left|\begin{pmatrix}3\\1\\-12\end{pmatrix}\cdot\begin{pmatrix}3\\1\\10\end{pmatrix}\right| = \sqrt{110}.$ *(M1)(A1)*

(ii) (a) $I_0(x) = \int_0^x e^{-t}dt = -e^{-t}|_0^x = 1 - e^{-x}.$ *(A1)*

(b) $I_n(x) = \int_0^x t^n e^{-t}\,dt$

$= \left[-t^n e^{-t}\right]_0^x + n\int_0^x t^{n-1}e^{-t}dt$ *(M2)*

$= -x^n e^{-x} + nI_{n-1}(x)$, as required. *(M1)AG*

(c) $I_3(x) = -x^3e^{-x} + 3I_2(x)$

$= -x^3e^{-x} + 3(-x^2e^{-x} + 2I_1(x))$

$= -x^3e^{-x} - 3x^2e^{-x} + 6(-xe^{-x} + I_0(x))$

$= -x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} + 6 - 6e^{-x}$

Therefore, $I_3(1) = 6 - 16e^{-1}$. *(M1) (A1)*

2. (a) $f(x) = \dfrac{1 + \ln x}{x}$

At A, $\ln x + 1 = 0 \Rightarrow x = e^{-1}$.

Therefore, $A = (e^{-1}, 0)$. *(M1) (A1)*

(b) $f'(x) = \dfrac{x\left(\frac{1}{x}\right) - (1 + \ln x)}{x^2} = \dfrac{-\ln x}{x^2}$, *(A2)*

and this is zero when $x = 1$.

Therefore, $B = (1, 1)$. *(M1) (A1)*

(c) $f''(x) = \dfrac{x^2\left(-\frac{1}{x}\right) + 2x\ln x}{x^4} = \dfrac{2\ln x - 1}{x^3}$, *(A2)*

and this is zero when $\ln x = \dfrac{1}{2}$ or $x = e^{1/2}$. *(A1)*

Clearly, $f''(x)$ changes sign at $x = e^{1/2}$, and so $C = (e^{1/2}, \frac{3}{2}e^{-1/2})$. *(M1) (A1)*

(d) The gradient of $(OT) = f'(x_0) = -\dfrac{\ln x_0}{x_0^2}$. *(M1)*

Therefore, the equation of (OT) is $y = \left(-\dfrac{\ln x_0}{x_0^2}\right)x$. *(A1)*

Note: At T, $f'(x_0) = \dfrac{f(x_0)}{x_0}$.

$\therefore \quad \dfrac{-\ln x_0}{x_0^2} = \dfrac{1 + \ln x_0}{x_0^2}$

Hence the equation of OT is $y = \left(\dfrac{1 + \ln x_0}{x_0^2}\right)x$.

Award *(M1) (C1)*

(e) Since T lies on both the curve and the tangent line,

$$\frac{1 + \ln x_0}{x_0} = \left(-\frac{\ln x_0}{x_0^2}\right)x_0.$$ *(R1)*

$$\Rightarrow \ln x_0 = -\frac{1}{2}.$$

Therefore, $x_0 = e^{-1/2}$. *(M1) (A1)*

(f) The x–coordinates are: $x_A = e^{-1}$, $x_T = e^{-1/2}$, $x_B = 1$, $x_C = e^{1/2}$.

Now, $\frac{x_T}{x_A} = \frac{x_B}{x_T} = \frac{x_C}{x_B} = e^{1/2}$, which is constant. *(M1)*

Therefore, **the x–coordinates are 4 consecutive terms of a GP, and the common ratio is $e^{1/2}$.** *(R1) (A1)*

(g) The required area = Area of ΔODT - Area of ADT

$$= \frac{1}{2}e^{-1/2}\left(\frac{1}{2}e^{1/2}\right) - \int_{e^{-1}}^{e^{-1/2}} \frac{1 + \ln x}{x}\,dx$$ *(A2)*

$$= \frac{1}{4} \left[\frac{1}{2}(1 + \ln x)^2\right]_{e^{-1}}^{e^{-1/2}}$$ *(M1) (A1)*

$$= \frac{1}{4} - \frac{1}{2}\left(\frac{1}{4} - 0\right)$$

$$= \frac{1}{8}$$ *(A1)*

3. (i) (a) In 6 experiments, p(exactly 2 successes) = $3p$(exactly 3 successes)

$$\Rightarrow \binom{6}{2}p^2(1-p)^4 = 3\binom{6}{3}p^3(1-p)^3$$ *(M2)*

$$\Rightarrow \quad 15p^2(1-p)^4 = 60p^3(1-p)^3$$

$$\Rightarrow \quad 1 - p = 4p$$

$$\Rightarrow \quad p = \frac{1}{5}$$ *(A2)*

(b) p(at least one success in n experiments) > 0.99

$\Rightarrow 1 - p$(no success in n experiments) > 0.99 *(M1)*

$$\Rightarrow \quad 1 - \left(\frac{4}{5}\right)^n > 0.99$$

$$\Rightarrow \quad \left(\frac{4}{5}\right)^n < 0.01$$ *(A1)*

$$\Rightarrow \quad 1.25^n > 100$$ *(A1)*

$$\Rightarrow \quad n > \frac{2}{\log 1.25} \approx 20.6$$ *(A1)*

Therefore, **the least number of times = 21.** *(A1)*

(ii) (a) Mean = np = 20×0.06 = **12**.

Standard deviation = $\sqrt{np(1-p)} = \sqrt{12 \times 0.94}$ = **3.36**. *(A2)*

(b) We require $p(X \geq 20)$ $= p\left(Z > \dfrac{19.5 - 12}{\sqrt{11.28}}\right)$ *(A2)*

$= p\,(Z > 2.233)$ *(A1)*

$= 1 - p\,(Z < 2.233)$ *(A1)*

$= 1 - 0.9872$

$=$ **0.0128** *(A1)*

4. (i) (a) $2z^2 - (2 - 2i)z - 5i = 0$

Therefore, $z = \dfrac{2 - 2i \pm \sqrt{(2 - 2i)^2 + 40i}}{4}$

$= \dfrac{2 - 2i \pm \sqrt{32i}}{4}$

$= \dfrac{1}{2}(1 - i) \pm \sqrt{2i}.$ *(M1) (A2)*

Without loss of generality, we may choose

$z_1 = \dfrac{1}{2}(1 - i) + \sqrt{2i}$ and $z_2 = \dfrac{1}{2}(1 - i) - \sqrt{2i}.$ *AG*

(b) Since $(2 - 2i)^2 = -8i$, [see part (a)], $(2 + 2i)^2 = 8i$ and so $(1 + i)^2 = 2i$. *(M1) (A1)*

Therefore, $z_1 = \dfrac{1}{2}(1 - i) + (1 + i)$ and

$z_2 = \dfrac{1}{2}(1 - i) - (1 + i).$

i.e. $z_1 = \dfrac{3}{2} + \dfrac{1}{2}i$ and $z_2 = -\dfrac{1}{2} - \dfrac{3}{2}i.$ *(M1) (A1)*

[or vice versa since $\sqrt{2i} = \pm(1 + i)$]

(ii) (a) If $z = \cos\theta + i\sin\theta$, then $z^n = \cos n\theta + i\sin n\theta$, and $z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$. (De Moivre) *(M1) (A1)*

Therefore, $z^n + \dfrac{1}{z^n} = 2\cos n\theta.$ *(M1)AG*

(b) $5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$

$\Rightarrow \quad 5z^2 - 11z + 16 - \frac{11}{z} + \frac{5}{z^2} = 0$ (since $z \neq 0$) *(M1) (R1)*

$\Rightarrow \quad 5\left(z^2 + \frac{1}{z^2}\right) - 11\left(z + \frac{1}{z}\right) + 16 = 0.$

* Let $z = \cos\theta + i\sin\theta$.

From part (a): $10\cos 2\theta - 22\cos\theta + 16 = 0$

$\Rightarrow \quad 10(2\cos^2\theta - 1) - 22\cos\theta + 16 = 0$

$\Rightarrow \quad 20\cos^2\theta - 22\cos\theta + 6 = 0$

$\rightarrow \quad 10\cos^2\theta - 11\cos\theta + 3 = 0$

$\Rightarrow \quad (5\cos\theta - 3)(2\cos\theta - 1) = 0$

$\Rightarrow \quad \cos\theta = \frac{3}{5}$ or $\cos\theta = \frac{1}{2}$. *(M2) (A2)*

Therefore, $\sin\theta = \pm\frac{4}{5}$ or $\sin\theta = \pm\frac{\sqrt{3}}{2}$.

This gives: $z = \frac{1}{5}(3 \pm 4i), \ \frac{1}{2}(1 \pm i\sqrt{3})$. *(A2)*

Alternative (from *):

$w = z + \frac{1}{z} \Rightarrow z^2 + \frac{1}{z^2} = w^2 - 2$

$5w^2 - 11w + 6 = 0 = (5w - 6)(w - 1)$

$\therefore z + \frac{1}{z} = 1$ or $\frac{6}{5}$

$z^2 - z + 1 = 0 \Rightarrow z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

or $5z^2 - 6z + 5 = 0 \Rightarrow z = \frac{3}{5} \pm \frac{4}{5}i$

Section B

5. (i) (a) $a * (x \otimes b) = c$

From the $*$ table: $a * b = c \Rightarrow x \otimes b = b$.

From the $\otimes$ table: $c \otimes b = b$.

Therefore, $x = c$. *(R3)*

(b) $(a \otimes x) * b = d$

From the $*$ table: $b * b = d \Rightarrow a \otimes x = b$.

From the $\otimes$ table: no x can be found for which $a \otimes x = b$.

Therefore, no x exists. *(R3)*

[Note: A student who consistently reads the tables in the form "top row element" $*$ (or $\otimes$) "left column element" may be awarded full marks for the answers (a) $x = a$ (b) $x = b$.]

(ii) (a) a^2ba^2b

$= a^2(ba^2)b$ (associativity)

$= a^2(ab)b$ (given)

$= a^3b^2$ (associativity)

$= ee$ (given)

$= e$ (identity) *(R3)*

(b) a^2ba

$= a(ab)a$ (associativity)

$= a(ba^2)a$ (given)

$= ab(a^3)$ (associativity)

$= abe$ (given)

$= ab$ (identity) *(R3)*

(iii) Let ϕ: $G \to H$ be the one–one, onto function such that

$\phi\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\right) = a + ib.$ *(M1)*

Thus $\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} \to a_1 + ib_1$ and $\begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} \to a_2 + ib_2$. *(A1)*

Now, $\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix}\begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 - b_1b_2 & -[a_1b_2 + a_2b_1] \\ a_1b_2 + a_2b_1 & a_1a_2 - b_1b_2 \end{pmatrix}$, *(A1)*

and $(a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$. *(A1)*

Therefore, $\phi\left(\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix}\begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix}\right) = (a_1 + ib_1)(a_2 + ib_2)$. *(A1)*

Thus, ***G* and *H* are isomorphic.** *(R1)*

(iv) (a) Z_4 has 4 members {0, 1, 2, 3} and Z_5 has 5 members {0, 1, 2, 3, 4}. *(M1)*

Therefore, $Z_4 \times Z_5$ has **4 × 5 = 20 members.** *(M1) (A1)*

(3, 2) * (1, 4) = (3 + 1 (mod 4), 2 + 4 (mod 5)) = **(0, 1)**. *(M1) (A1)*

(b) $Z_2 \times Z_3$ = {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)} *(M1)*

Now, $(1, 1)^2 = (0, 2)$; $(1, 1)^3 = (1, 0)$; $(1, 1)^4 = (0, 1)$;
$(1, 1)^5 = (1, 2)$; $(1, 1)^6 = (0, 0)$ which is the identity.

Therefore, $Z_2 \times Z_3$ is cyclic. *(R2)*

Both (1, 1) and (1, 2) are generators. *(A2)*

$[(0, 1)^3 = (0, 0)$; $(0, 2)^3 = (0, 0)$; $(1, 2)^2 = (0, 0)]$ *(R1)*

(c) $Z_2 \times Z_4$ = {(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3)} *(M1)*

$(0, 1)^2 = (0, 2)$; $(0, 1)^3 = (0, 3)$; $(0, 1)^4 = (0, 0)$

$(0, 3)^2 = (0, 2)$; $(0, 3)^3 = (0, 1)$; $(0, 3)^4 = (0, 0)$

$(1, 3)^2 = (0, 2)$; $(1, 3)^3 = (1, 1)$; $(1, 3)^4 = (0, 0)$

$(1, 1)^2 = (0, 2)$; $(1, 1)^3 = (1, 3)$; $(1, 1)^4 = (0, 0)$

$(0, 2)^2 = (0, 0)$. $(1, 2)^2 = (0, 0)$. $(1, 0)^2 = (0, 0)$

Therefore, (0, 1), (0, 3), (1, 1) and (1, 3) all have order 4. *(R4) (A2)*

6. (i) (a)

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

***(A2)* (-1 per error to a max. of two).**

***(A2)* (-1 per error to a max. of two).**

(b)

$$A^2 = \begin{pmatrix} 3 & 2 & 3 & 3 & 3 \\ 2 & 4 & 4 & 2 & 4 \\ 3 & 4 & 5 & 3 & 5 \\ 3 & 2 & 3 & 3 & 3 \\ 3 & 4 & 5 & 3 & 5 \end{pmatrix}$$

***(A2)* (-1 per error to a max. of two).**

An entry a_{ij} in A^2 is the **number of different walks between the vertices v_i and v_j which passes through exactly one vertex v_k which is the same as the number of paths of length 2 between v_i and v_j.** *(A1)*

(c) The sum of the entries in the jth column of A gives the **degree of the vertex v_j** provided there is no loop at that vertex. If there is one or more loops at vertex v_j, then **$\deg(v_j)$ = jth col. sum − 1 + 2(no. of loops at v_j)**, since each loop contributes 2 to the degree of the vertex. *(R1) (A1)*

Note: Also accept $\deg(v_j)$ = jth col.sum + (no. of loops at v_j).

(d) **The column sum is 2 for each edge, since it joins exactly two vertices. The column sum is 1 for a loop.** *(R1) (A1)*

(ii) (a) In any loop-free, undirected graph, the maximum number of edges is $\binom{v}{2}$. *(A1)*

Hence, $e \le \binom{v}{2} = \dfrac{v(v-1)}{2} \Rightarrow 2e \le v^2 - v$. *(A3)AG*

(b) In the loop-free, directed case, $e \le v(v-1) = v^2 - v$. *(A2)*

(iii) (a) Consider one vertex. There are $(n - 1)$ possible edges to start the path. Once we are at the second vertex, we have $(n - 2)$ possible edges; at the third vertex, $(n - 3)$ edges; at the fourth vertex, $(n - 4)$ edges. This gives $(n - 1)(n - 2)(n - 3)(n - 4)$ paths. Now, we have n vertices to choose as our starting vertex, and, according to the hint, we have counted each possible path twice.

The required no. of paths $= \dfrac{n(n - 1)(n - 2)(n - 3)(n - 4)}{2}$. *(R1) (A1)*

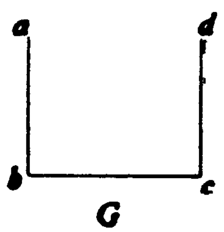
(b) Let e_1 be the number of edges in G, a simple undirected graph, and let e_2 be the number of edges in $\overline{G}$.

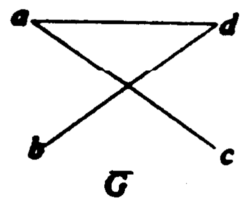
Then $e_1 + e_2 = \binom{n}{2} = \dfrac{n(n - 1)}{2}$, the number of edges in k_n.

Since G is self–complementary, $e_1 = e_2$, so

$e_1 = \dfrac{1}{2}\binom{n}{2} = \dfrac{n(n - 1)}{4}$. *(R2) (A2)*

(c) The required example is:





The isomorphism from G to $\overline{G}$ is as follows:

element	a	b	c	d
isomorphic image	b	d	a	c

(A2)

(d) Since G is self-complementary, it has $\frac{1}{2}\binom{n}{2} = \frac{n^2 - n}{4}$ edges.

Therefore, $4 \mid n(n - 1)$.

Now, if n is even, $n - 1$ is odd and so $4 \mid n$.

i.e. $n = 4k, \ k \in N^*$. *(R2)AG*

If $(n - 1)$ is even, n is odd, and so $4 \mid (n - 1)$.

i.e. $n - 1 = 4k$ or $n = 4k + 1, \ k \in N^*$. *(R2)AG*

(iv) Without loss of generality, let the vertices and their degrees be:

vertex	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*
degree	1	1	1	2	3	4	5	7

Vertex h has degree 7 and so must be connected to each of the other 7 vertices. Thus, a, b, c cannot be connected to any other vertex other than h. Vertex g has degree 5. It is connected already to h, and so must be connected to four other vertices. But there are only 3 (d, e, f) available. Therefore, **no graph can be drawn.** *(R3) (A2)*

(v)

counter	vertex used	vertex added	weight added	cumulative
1		*e*	0	0
2	*e*	*h*	1	1
3	*e, h*	*d*	2	3
4	*d, e, h*	*f*	2	5
5	*d, e, f, h*	*b*	3	8
6	*b, d, e, f, h*	*a*	2	10
7	*a, b, d, e, f, h*	*c*	2	12
8	*a, b, c, d, e, f, h*	*g*	3	15
9	*a, b, c, d, e, f, g, h*	*i*	3	18
10	*a, b, c, d, e, f, g, h, i*		0	18

Therefore, **the minimum cost = $18 m.** *(M5) (A1)*

7. (i) The Poisson distribution with mean μ is defined by $p(x) = \dfrac{\mu^x e^{-x}}{x!}$ where x represents the number of arrivals during a given time interval.

(a) The required probability $= p(0) = \dfrac{3^0 e^{-3}}{0!} = e^{-3} = \mathbf{0.0498}$. *(M3) (A1)*

(b) The required probability
$= p(1) + p(2) = \dfrac{3e^{-3}}{1!} + \dfrac{3^2 e^{-3}}{2!} = \mathbf{0.373}$. *(M3) (A1)*

(ii) A t-distribution with $\bar{x} = 83.7$, $s = 12.9$, $n = 9$ and 8 degrees of freedom is appropriate here. *(R1) (A1)*

Since the standard deviation of the population is unknown, the standard deviation of $\bar{x}$ is estimated with $\dfrac{s}{\sqrt{n}} = \dfrac{12.9}{3} = 4.3$. *(R1) (A1)*
The critical value associated with a 95% confidence interval in this two-tailed test is $t = 2.306$. *(A2)*

Hence, the required interval $= \left(\bar{x} - t\dfrac{s}{\sqrt{n}},\ \bar{x} + t\dfrac{s}{\sqrt{n}}\right) = \mathbf{(73.8, 93.6)}$. *(M2) (A2)*

Note: Some candidates will use instead of $\dfrac{s}{\sqrt{n}}, \dfrac{s}{\sqrt{n-1}}$, by using the unbiased estimator of population variance, viz. $s^2\left(\dfrac{n}{n-1}\right)$ in place of s^2. Then they will get the following:

Standard deviation of $\bar{x}$ is estimated with
$\dfrac{s}{\sqrt{n-1}} = \dfrac{12.9}{\sqrt{8}} = 4.56$ *(R1) (A1)*

The critical value of t here is 2.306. *(A2)*

Hence, the required interval is $\left(\bar{x} - t\dfrac{s}{\sqrt{n-1}},\ \bar{x} + \dfrac{t}{\sqrt{n-1}}\right)$

$= (73.2, 94.2)$ *(M2) (A2)*

(iii) $H_0 : p \geq 0.4$ *(A1)*

$H_1 : p < 0.4$

The sample proportion is $p = \frac{25}{100} = 0.25$ with mean $\mu = 0.4$ and *(A1)*

standard deviation $\sigma = \sqrt{\frac{(0.4)(0.6)}{100}} = 0.049$. *(A2)*

If we assume a Normal distribution, then the corresponding z–value is

$z = \frac{0.25 - 0.4}{0.049} = -3.062$. *(R1) (A2)*

Now, $p(Z < -3.062) = 0.0011$ which is less than 0.01. *(A2)*

Therefore, **the claim is rejected.** *(A1)*

(iv) Let μ_A be the population mean for tyre A and μ_B that for tyre B. Since the sample size = 10, the t–distribution is appropriate.

The number of degrees of freedom = 9. *(R2)*

(a) For a 95% confidence interval, the appropriate value of $t = 2.262$. *(A2)*

Hence, $\mu_A - \mu_B = 2000 \pm 2.262\left(\frac{2000}{\sqrt{10}}\right) = 2000 \pm 1430.6$ miles. *(R1) (A3)*

Therefore, $\mathbf{569 \leq \mu_A - \mu_B \leq 3430}$ **miles** *(A2)*

Note: Some candidates will use the unbiased estimator of population variance viz. $s^2\left(\frac{n}{n-1}\right)$ to estimate the variance of $\bar{x}$. Then they will have *(R1) (A3)*

$\mu_A - \mu_B = 2000 + 2.262\left(\frac{2000}{\sqrt{9}}\right) = 2000 \pm 1508$

$\therefore$ $\mathbf{492 \leq \mu_A - \mu_B \leq 3508}$ **miles**. *(A2)*

(b) We can accept H_0 if $\mu_A - \mu_B = 0$ falls in the 95% confidence interval. But this is clearly not so. Therefore, H_0 **is rejected.** *(R1) (A1)*

8. (i) (a) If $e^x = x^4$, then $x = 4\ln x$.

Let $f(x) = 4\ln x$, then $f'(x) = \frac{4}{x}$. *(A1)*

Thus, $|f'(8)| = 0.5 < 1$, and so **the given formula may be used.**

If $e^x = x^4$, then $x = e^{x/4}$. *(R1) (M1)*

Let $f(x) = e^{x/4}$, then $f'(x) = \frac{1}{4}e^{x/4}$. *(A1)*

Thus, $|f'(1)| = \frac{1}{4}e^{1/4} \approx 0.32 < 1$, and so **the formula may be used.** *(R1) (M1)*

i	x_i	i	x_i
0	8	0	1
1	8.3178	1	1.2840
2	8.4736	2	1.3785
3	8.5478	3	1.4115
4	8.5827	4	1.4231
5	8.5990	5	1.4273
6	8.6066	6	1.4288
7	8.6101	7	1.4293
8	8.6118	8	1.4295
9	8.6125	9	1.4296
10	8.6129	10	1.4296
11	8.6130		
12	8.6131		
13	8.6131		

The required roots are 8.613 and 1.430 (3 dec pl.). *(A2)*

(b) $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ where $f(x) = e^x - x^4$, and $f'(x) = e^x - 4x^3$. *(M1) (A1)*

Thus, $x_1 = -1 - \left(\dfrac{e^{-1} - 1}{e^{-1} + 4}\right) = -1 - \left(\dfrac{1 - e}{1 + 4e}\right) = \dfrac{\mathbf{-2 - 3e}}{\mathbf{1 + 4e}}$. *(A2)*

(c) The area $= \int_0^1 (e^x - x^4)\,dx = \left[e^x - \dfrac{1}{5}x^5\right]_0^1 = \mathbf{e - 1.2}$. *(M1) (A2)*

(d) The area $\approx \dfrac{h}{3}(f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1))$, *(M2)*

$h = 0.25$.

Thus, **the area = 1.5178 (4 dec. pl.)** *(A1)*

(e) The error term is $-\dfrac{(b - a)^5 f^{(4)}(c)}{180n^4}$ where $c \in [a, b]$. *(A3)*

The error term is $-\dfrac{1^5(e^c - 24)}{(180)(4^4)} \le \dfrac{23}{(180)(256)} < 0.0005$, since the minimum value of e^c is $e^0 = 1$. *(R1) (A1)*

Therefore, **the maximum error = 0.0005.** *(A1)*

Since estimated value + maximum error = 1.5178 + 0.0005 = 1.5183, and e − 1.2 = 1.5183 (4 dec. pl.), **the answers to parts (c) and (d) are consistent with the calculated maximum error.** *(A1)*

(f) We require the value of n for which $\dfrac{23}{180n^4} < 5 \times 10^{-6}$. *(A1)*

Thus, $n^4 > \dfrac{23}{900} \times 10^6 \Rightarrow n > 12.6$. *(A2)*

Therefore, **the required value of *n* is 14.** (*n* must be even) *(R1) (A1)*

(ii) (a) Let $f(x) = \dfrac{1}{x(\ln x)^2}$, $x \geq 2$. It is obvious that $f(x)$ is continuous on $[2, \infty)$ since $x(\ln x)^2$ is continuous and non-zero on $[2, \infty)$. *(R1)*

Clearly, $f(x) > 0$ for $x \in [2, \infty)$. *(A1)*

Now, $f'(x) = -\dfrac{2x(\ln x)\frac{1}{x} + (\ln x)^2}{(x(\ln x)^2)^2} = -\dfrac{2 + \ln x}{x^2(\ln x)^3} < 0$ for $x \geq 2$. *(A1)*

Therefore, $f(x)$ is decreasing on $[2, \infty)$ and so $f(x)$ satisfies the conditions under which the Integral Test may be used. *(A1)*

$$\text{Now, } \int_2^\infty \frac{dx}{x(\ln x)^2} = \lim_{b \to \infty} \int_2^b \frac{dx}{x(\ln x)^2}$$

$$= \lim_{b \to \infty} \left[-\frac{1}{\ln x}\right]_2^b$$

$$= \lim_{b \to \infty} \left(\frac{1}{\ln 2} - \frac{1}{\ln b}\right)$$

$$= \frac{1}{\ln 2} < \infty.$$

Therefore, $\sum_{n=2}^{\infty} \dfrac{1}{n(\ln n)^2}$ converges. *(M1) (A1)*

(b) Let $a_n = \dfrac{(-1)^n}{n}$. Then the series $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \dfrac{1}{n}$ is divergent. *(A1)*

Also, the sequence $\{|a_n|\}$ is decreasing since

$\dfrac{1}{n+1} - \dfrac{1}{n} = \dfrac{-1}{n(n+1)} < 0$, and $\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} \dfrac{1}{n} = 0$. *(R1)*

Thus, $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n}$ converges. *(A1)*

Therefore, $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n}$ converges conditionally. *(A1)AG*