INTERNATIONAL BACCALAUREATE
BACCALAURÉAT INTERNATIONAL
BACHILLERATO INTERNACIONAL

MATHEMATICAL METHODS

Standard Level

Friday 5 November 1999 (morning)

Paper 2 — 2 hours

This examination paper consists of 2 sections, Section A and Section B.
Section A consists of 4 questions.
Section B consists of 2 questions.
The maximum mark for Section A is 80.
The maximum mark for Section B is 40.
The maximum mark for this paper is 120.

INSTRUCTIONS TO CANDIDATES

Do NOT open this examination paper until instructed to do so.

Answer all FOUR questions from Section A and ONE question from Section B.

Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures as appropriate.

EXAMINATION MATERIALS

Required:
IB Statistical Tables
Millimetre square graph paper
Calculator
Ruler and compasses

Allowed:
A simple translating dictionary for candidates not working in their own language

FORMULAE

Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$

Arithmetic series: $S_n = \dfrac{n}{2}\{2a + (n-1)d\}$

Geometric series: $S_n = \dfrac{a(r^n - 1)}{r - 1}, \quad r \neq 1$

Arc length of a circle: $s = r\theta$

Area of a sector of a circle: $A = \frac{1}{2}r^2\theta$

Area of a triangle: $A = \frac{1}{2}ab\sin C$

Statistics: If $(x_1, x_2, \ldots, x_n)$ occur with frequencies $(f_1, f_2, \ldots, f_n)$ then the mean m and standard deviation s are given by

$$m = \frac{\Sigma f_i x_i}{\Sigma f_i} \qquad s = \sqrt{\frac{\Sigma f_i (x_i - m)^2}{\Sigma f_i}}, \qquad i = 1, 2, \ldots, n$$

Newton-Raphson formula: (For finding a root of $f(x) = 0$)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Integration by parts: (Analytical Geometry and Further Calculus Option only)

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

A correct answer with **no** *indication of the method used will normally receive* **no** *marks. You are therefore advised to show your working.*

SECTION A

Answer all FOUR questions from this section.

1. *[Maximum mark: 16]*

The diagram shows the parallelogram $PQRS$ formed by the four lines (PQ), (QR), (RS) and (SP) .

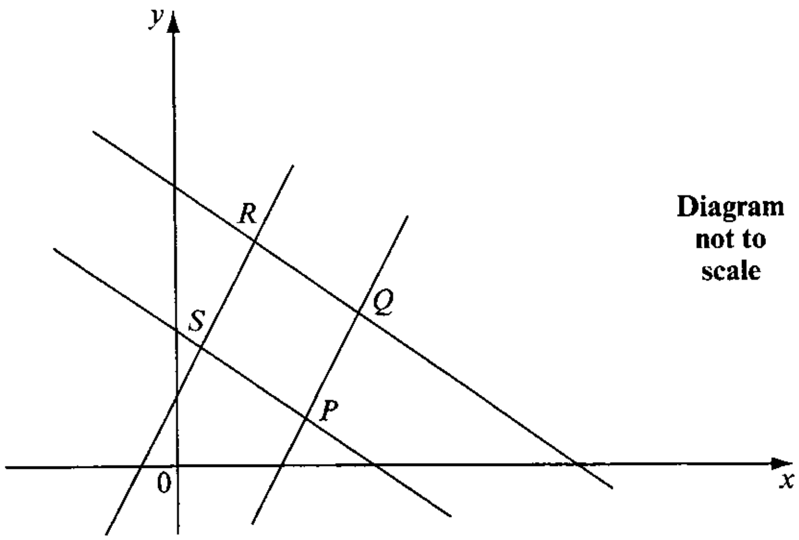


Diagram not to scale

The line (SP) has equation $6x + 8y = 39$ and the line (PQ) has equation $8x - 6y = 27$.

(a) Show that at the point P , $x = 4.5$, and find the value of y . *[4 marks]*

The point R has coordinates $(3.5 , 8.5)$.

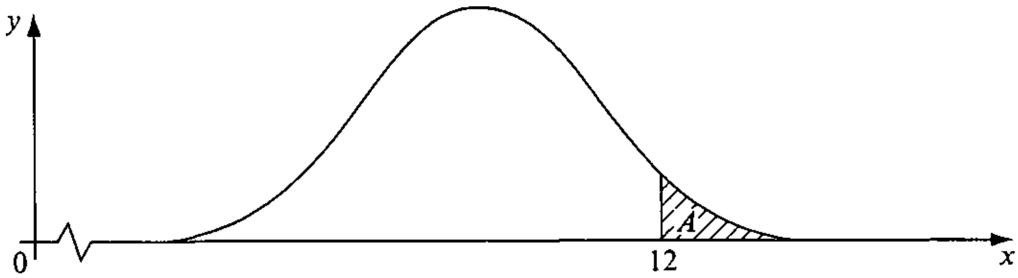
(b) Find an equation for the line (RQ) . Hence show that at the point Q , $x = 7.5$, and find the value of y . *[4 marks]*

(c) Find $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{PQ} \cdot \overrightarrow{QR}$. *[4 marks]*

(d) Hence, or otherwise, show that $PQRS$ is a square. *[4 marks]*

2. *[Maximum mark: 16]*

The graph shows a normal curve for the random variable X, with mean μ and standard deviation σ.



It is known that $p(X \geqslant 12) = 0.1$.

(a) The shaded region A is the region under the curve where $x \geqslant 12$. Write down the area of the shaded region A . *[1 mark]*

It is also known that $p(X \leqslant 8) = 0.1$.

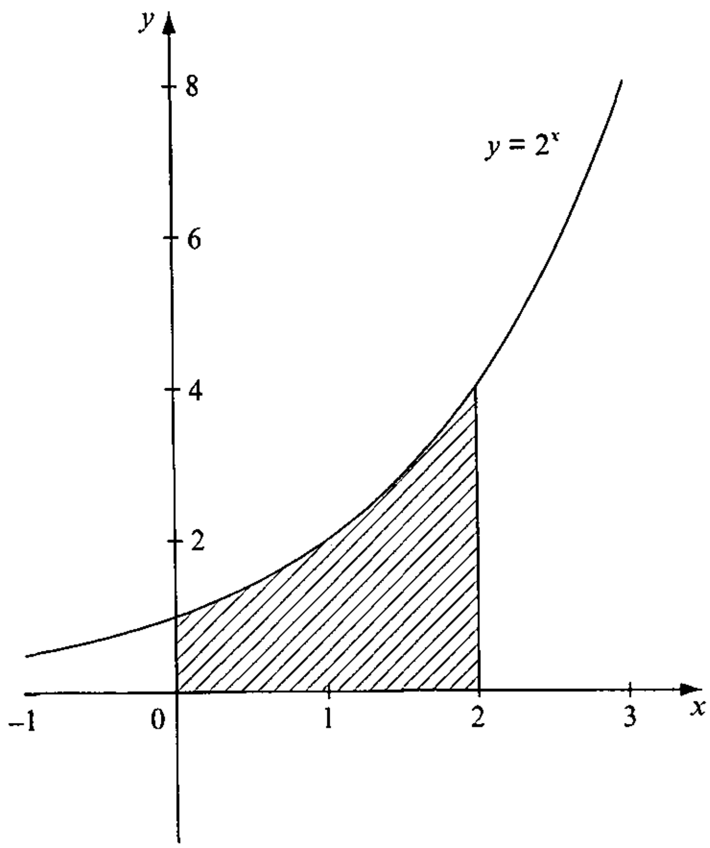
(b) Find the value of μ, explaining your method in full. *[5 marks]*

(c) Show that $\sigma = 1.56$ to an accuracy of three significant figures. *[5 marks]*

(d) Find $p(X \leqslant 11)$. *[5 marks]*

3. *[Maximum mark: 24]*

The diagram shows the graph of $y = 2^x$ for $-1 \leqslant x \leqslant 3$.



(a) Use the trapezium rule with **four intervals** to approximate $\int_0^2 2^x \, dx$. *[5 marks]*

(b) Explain why the result in (a) must over-estimate the value of the integral. *[2 marks]*

(c) Show that the curve $y = 1 + 0.5x + 0.5x^2$ intersects the curve $y = 2^x$ at $x = 0$, $x = 1$ and $x = 2$. *[4 marks]*

(d) Find $\int (1 + 0.5x + 0.5x^2) \, dx$. **Hence** find $\int_0^2 (1 + 0.5x + 0.5x^2) \, dx$. *[5 marks]*

(e) Use the identity $2^x = e^{x \ln 2}$ to show that $\int 2^x dx = \frac{2^x}{\ln 2} + C$. **Hence** find the exact value of $\int_0^2 2^x dx$. *[5 marks]*

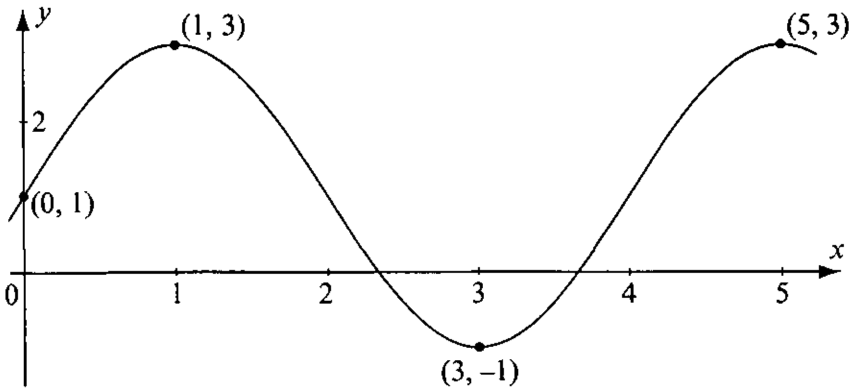
(f) Use your answers to parts (d) and (e) to find the relative error when $\int_0^2 (1 + 0.5x + 0.5x^2) \, dx$ is used as an approximation to $\int_0^2 2^x dx$. *[3 marks]*

4. *[Maximum mark: 24]*

The diagram shows the graph of the function f given by

$$f(x) = A \sin\left(\frac{\pi}{2}x\right) + B,$$

for $0 \leqslant x \leqslant 5$, where A and B are constants, and x is measured in radians.



The graph includes the points (1 , 3) and (5 , 3) , which are maximum points of the graph.

(a) Write down the values of $f(1)$ and $f(5)$. *[2 marks]*

(b) Show that the period of f is 4 . *[2 marks]*

The point (3 , –1) is a minimum point of the graph.

(c) Show that $A = 2$, and find the value of B . *[5 marks]*

(d) Show that $f'(x) = \pi \cos\left(\frac{\pi}{2}x\right)$. *[4 marks]*

The line $y = k - \pi x$ is a tangent line to the graph for $0 \leqslant x \leqslant 5$.

(e) Find

(i) the point where this tangent meets the curve;

(ii) the value of k . *[6 marks]*

(f) Solve the equation $f(x) = 2$ for $0 \leqslant x \leqslant 5$. *[5 marks]*

SECTION B

Answer ONE question from this section.

Analytical Geometry and Further Calculus

5. *[Maximum mark: 40]*

(i) The line-segment $[PR]$ has end-points $P(-8\,,\,2)$ and $R(14\,,\,6)$.

(a) Find the mid-point of $[PR]$. *[2 marks]*

(b) The equation of the circle with $[PR]$ as diameter is

$$x^2 + y^2 - 6x + Ay + B = 0$$

Find the values of A and B. *[5 marks]*

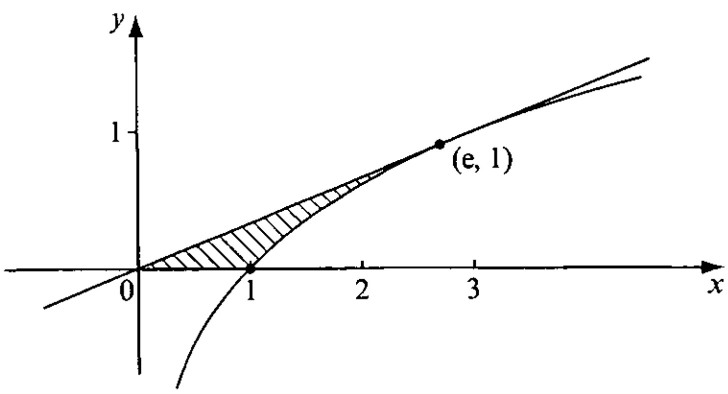
(c) (i) Verify that the coordinates of the point Q(8 , 14) satisfy the equation of the circle.

(ii) Show that $P\widehat{Q}R$ is a right-angle. *[3 marks]*

(ii) (a) Find the equation of the tangent line to the curve $y = \ln x$ at the point (e , 1), and verify that the origin is on this line. *[4 marks]*

(b) Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}(x \ln x - x) = \ln x$. *[2 marks]*

(c) The diagram shows the region enclosed by the curve $y = \ln x$, the tangent line in part (a), and the line $y = 0$.



Use the result of part (b) to show that the area of this region is $\frac{1}{2}\mathrm{e} - 1$. *[4 marks]*

(*This question continues on the following page*)

(*Question 5 continued*)

(iii) A curve has equation $y = x(x-4)^2$.

(a) For this curve find

(i) the x-intercepts;

(ii) the coordinates of the maximum point;

(iii) the coordinates of the point of inflexion. *[9 marks]*

(b) Use your answers to part (a) to sketch a graph of the curve for $0 \leqslant x \leqslant 4$, clearly indicating the features you have found in part (a). *[3 marks]*

(c) (i) On your sketch indicate by shading the region whose area is given by the following integral:

$$\int_0^4 x(x-4)^2 \, dx .$$

(ii) Explain, using your answer to part (a), why the value of this integral is greater than 0 but less than 40 . *[3 marks]*

(d) Use the method of integration by substitution with $u = (x-4)$ to evaluate the integral. *[5 marks]*

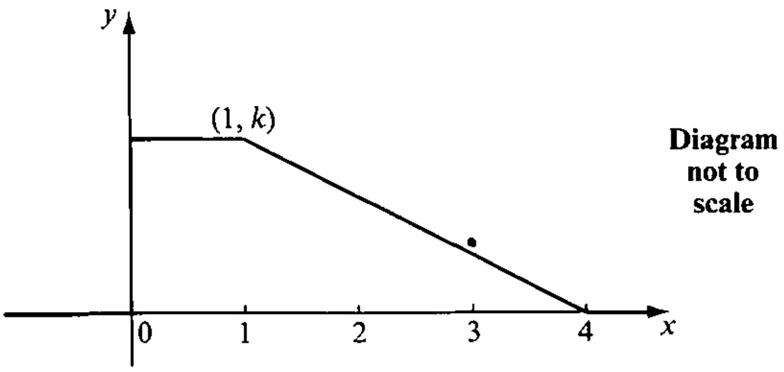
Further Probability and Statistics

6. *[Maximum mark: 40]*

(i) The continuous random variable X has probability density function f, where

$$f(x) = k\,, \qquad 0 \leqslant x \leqslant 1\,,$$
$$f(x) = a - bx\,, \qquad 1 \leqslant x \leqslant 4\,,$$
$$f(x) = 0\,, \qquad \text{otherwise.}$$

The following diagram represents the graph of f.



(a) Show that $k = 0.4$. *[3 marks]*

(b) Show that $b = \frac{2}{15}$, and find the value of a. *[3 marks]*

(c) Find

(i) $p(X \geqslant 2)$;

(ii) $p(0.5 \leqslant X \leqslant 2)$. *[5 marks]*

(d) Find $E(X)$. *[5 marks]*

(e) Show that the median is 1.26 to three significant figures. *[4 marks]*

(This question continues on the following page)

(Question 6 continued)

(ii) A fair coin is tossed eight times. Calculate

(a) the probability of obtaining exactly 4 heads; *[2 marks]*

(b) the probability of obtaining exactly 3 heads; *[1 mark]*

(c) the probability of obtaining 3 , 4 or 5 heads. *[3 marks]*

The discrete random variable R represents the number of heads obtained in 64 tosses of a fair coin.

(d) Find the mean and standard deviation of R . *[4 marks]*

(e) Use a normal approximation to the binomial distribution to find $p(24 \leqslant R \leqslant 40)$. *[5 marks]*

When another coin is tossed 64 times, 40 heads are obtained. As a result, it is suspected that this coin has a bias towards heads.

(f) Explain whether, at the 5% level, the results of this trial confirm the suspicion. *[5 marks]*