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FURTHER MATHEMATICS
STANDARD LEVEL
PAPER 2

Monday 7 May 2012 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics SL*** *information booklet* is required for this paper.
- The maximum mark for this examination paper is *[120 marks]*.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Total mark: 23]*

Part A *[Maximum mark: 11]*

The area of an equilateral triangle is 1 cm^2. Determine the area of

(a) the circumscribed circle; *[8 marks]*

(b) the inscribed circle. *[3 marks]*

Part B *[Maximum mark: 12]*

The points A, B have coordinates $(1, 0), (0, 1)$ respectively. The point $P(x, y)$ moves in such a way that $AP = kBP$ where $k \in \mathbb{R}^+$.

(a) When $k = 1$, show that the locus of P is a straight line. *[3 marks]*

(b) When $k \neq 1$, the locus of P is a circle.

(i) Find, in terms of k, the coordinates of C, the centre of this circle.

(ii) Find the equation of the locus of C as k varies. *[9 marks]*

2. *[Total mark: 25]*

Part A *[Maximum mark: 16]*

The graph H has the following adjacency matrix.

$$\begin{array}{c c} & \begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(a) (i) Show that H is bipartite.

(ii) Draw H as a planar graph. *[3 marks]*

(b) (i) Explain what feature of H guarantees that it has an Eulerian circuit.

(ii) Write down an Eulerian circuit in H. *[3 marks]*

(c) (i) Find the number of different walks of length five joining A to B.

(ii) Determine how many of these walks go through vertex F after passing along two edges. *[6 marks]*

(d) Find the maximum number of extra edges that can be added to H while keeping it simple, planar and bipartite. *[4 marks]*

Part B *[Maximum mark: 9]*

(a) Find the smallest positive integer m such that $3^m \equiv 1 \pmod{22}$. *[2 marks]*

(b) Given that $3^{49} \equiv n \pmod{22}$ where $0 \le n \le 21$, find the value of n. *[4 marks]*

(c) Solve the equation $3^x \equiv 5 \pmod{22}$. *[3 marks]*

3. *[Maximum mark: 29]*

(a) (i) Show that $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)\right) = 2\sec^3\theta$.

(ii) Hence write down $\int \sec^3\theta \,\mathrm{d}\theta$. *[5 marks]*

(b) Consider the differential equation $(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + xy = 1 + x^2$ given that $y = 1$ when $x = 0$.

(i) Use Euler's method with a step length of 0.1 to find an approximate value for y when $x = 0.3$.

(ii) Find an integrating factor for determining the exact solution of the differential equation.

(iii) Find the solution of the equation in the form $y = f(x)$.

(iv) To how many significant figures does the approximation found in part (i) agree with the exact value of y when $x = 0.3$? *[24 marks]*

4. *[Total mark: 25]*

Part A *[Maximum mark: 12]*

The function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$ is defined by $\boldsymbol{X} \mapsto \boldsymbol{AX}$, where $\boldsymbol{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c, d are all non-zero.

(a) Show that f is a bijection if $\boldsymbol{A}$ is non-singular. *[7 marks]*

(b) Suppose now that $\boldsymbol{A}$ is singular.

(i) Write down the relationship between a, b, c, d.

(ii) Deduce that the second row of $\boldsymbol{A}$ is a multiple of the first row of $\boldsymbol{A}$.

(iii) Hence show that f is not a bijection. *[5 marks]*

(This question continues on the following page)

(Question 4 continued)

Part B *[Maximum mark: 13]*

Consider the group $\{S, +_m\}$ where $S = \{0, 1, 2 \ldots m-1\}$, $m \in \mathbb{N}$, $m \geq 3$ and $+_m$ denotes addition modulo m.

(a) Show that $\{S, +_m\}$ is cyclic for all m. *[3 marks]*

(b) Given that m is prime,

(i) explain why all elements except the identity are generators of $\{S, +_m\}$;

(ii) find the inverse of x, where x is any element of $\{S, +_m\}$ apart from the identity;

(iii) determine the number of sets of two distinct elements where each element is the inverse of the other. *[7 marks]*

(c) Suppose now that $m = ab$ where a, b are unequal prime numbers. Show that $\{S, +_m\}$ has two proper subgroups and identify them. *[3 marks]*

5. *[Maximum mark: 18]*

(a) The continuous random variable X takes values only in the interval $[a, b]$ and F denotes its cumulative distribution function. Using integration by parts, show that:

$$E(X) = b - \int_a^b F(x)\,\mathrm{d}x\,.$$ *[4 marks]*

(b) The continuous random variable Y has probability density function f given by:

$$f(y) = \cos y, \quad 0 \le y \le \frac{\pi}{2}$$
$$f(y) = 0, \quad \text{elsewhere}\,.$$

(i) Obtain an expression for the cumulative distribution function of Y, valid for $0 \le y \le \frac{\pi}{2}$. Use the result in (a) to determine $E(Y)$.

(ii) The random variable U is defined by $U = Y^n$, where $n \in \mathbb{Z}^+$. Obtain an expression for the cumulative distribution function of U valid for $0 \le u \le \left(\frac{\pi}{2}\right)^n$.

(iii) The medians of U and Y are denoted respectively by m_u and m_y. Show that $m_u = m_y^n$. *[14 marks]*