International Baccalaureate®
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MARKSCHEME

May 2009

FURTHER MATHEMATICS

Standard Level

Paper 2

16 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) are often dependent on the preceding ***M*** mark.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.* do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question. Award the marks as usual and then write –1(****MR****) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled ***(d)*** *and a brief* ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates may not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

Candidates should be penalized ***once only IN THE PAPER*** *for an accuracy error* ***(AP)****. Award the marks as usual then write* ***(AP)*** *against the answer. On the* **front** *cover write –1****(AP)****. Deduct 1 mark from the total for the paper, not the question*.

- If a final correct answer is incorrectly rounded, apply the ***AP***.
- If the level of accuracy is not specified in the question, apply the ***AP*** for correct answers not given to three significant figures.

If there is no working shown, and answers are given to the correct two significant figures, apply the ***AP***. However, do **not** accept answers to one significant figure without working.

1. Part A

(a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

$e^{-\frac{x^2}{2}} = 1 + \left(-\frac{x^2}{2}\right) + \frac{\left(-\frac{x^2}{2}\right)^2}{2!} + \frac{\left(-\frac{x^2}{2}\right)^3}{3!} + \frac{\left(-\frac{x^2}{2}\right)^4}{4!} + \ldots$ ***M1A1***

$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}}\left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{384}\right)$ ***A1***

[3 marks]

(b) (i) $\frac{1}{\sqrt{2\pi}}\int_0^x 1 - \frac{t^2}{2} + \frac{t^4}{8} - \frac{t^6}{48} + \frac{t^8}{384}\,\mathrm{d}t$ ***M1***

$= \frac{1}{\sqrt{2\pi}}\left(x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} + \frac{x^9}{3456}\right)$ ***A1***

$\mathrm{P}(Z \le x) = 0.5 + \frac{1}{\sqrt{2\pi}}\left(x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} + \frac{x^9}{3456} - \ldots\right)$ ***R1A1***

(ii) $\mathrm{P}(-0.5 \le Z \le 0.5) = \frac{2}{\sqrt{2\pi}}\left(0.5 - \frac{0.5^3}{6} + \frac{0.5^5}{40} - \frac{0.5^7}{336} + \frac{0.5^9}{3456} - \ldots\right)$ ***M1***

$= 0.38292 = 0.383$ ***A1***

[6 marks]

Sub-total [9 marks]

continued …

Question 1 continued

Part B

(a) this is a two tailed test of the sample mean $\bar{x}$
we use the central limit theorem to justify assuming that ***R1***

$\bar{X} \sim \mathrm{N}\left(28, \frac{0.54^2}{24}\right)$ ***R1A1***

$\mathrm{H}_0 : \mu = 28$ ***A1***

$\mathrm{H}_1 : \mu \neq 28$ ***A1***

[5 marks]

(b) since $\mathrm{P}(\text{Type I error}) = 0.035$, critical value 2.108 ***(M1)A1***

and $\left(\bar{x} \le 28 - 2.108\sqrt{\frac{0.54^2}{24}} \text{ or } \bar{x} \ge 28 + 2.108\sqrt{\frac{0.54^2}{24}}\right)$ ***(M1)(A1)(A1)***

$\bar{x} \le 27.7676$ or $\bar{x} \ge 28.2324$
so $\bar{x} \le 27.8$ or $\bar{x} \ge 28.2$ ***A1A1***

[7 marks]

(c) if $\mu = 28.1$

$\bar{X} \sim \mathrm{N}\left(28.1, \frac{0.54^2}{24}\right)$ ***R1***

$\mathrm{P}(\text{Type II error}) = \mathrm{P}(27.7676 < \bar{x} < 28.2324)$
$= 0.884$ ***A1***

Note:	Depending on the degree of accuracy used for the critical region the answer for part (c) can be anywhere from 0.8146 to 0.879.

[2 marks]

Sub-total [14 marks]

Total [23 marks]

2. Part A

(a) (i)

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

A2

Note: Award ***A1*** for table if exactly one error and ***A0*** if more than one error.

all elements belong to $\mathbb{Z}_4$ so it is closed ***A1***
0 is the identity element ***A1***
2 is self inverse ***A1***
1 and 3 are an inverse pair ***A1***
hence every element has an inverse
hence $\{\mathbb{Z}_4, +_4\}$ form a group *G* ***AG***

(ii) $1 +_4 1 \equiv 2 (\text{mod } 4)$
$1 +_4 1 +_4 1 \equiv 3 (\text{mod } 4)$
$1 +_4 1 +_4 1 +_4 1 \equiv 0 (\text{mod } 4)$ ***M1A1***
hence 1 is a generator ***R1***
therefore *G* is cyclic ***AG***
(3 is also a generator)

[9 marks]

continued …

Question 2 Part A continued

(b)

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

A1A1

EITHER

for the group $(\{1, 2, 3, 4\}, \times_5)$
1 is the identity and 4 is self inverse ***A1***
2 and 3 are an inverse pair ***A1***

OR

for G,	for H,
0 has order 1	1 has order 1
1 has order 4	2 has order 4
2 has order 2	3 has order 4
3 has order 4	4 has order 2

A1A1

THEN

hence there is a bijection ***R1***
$h(1) \to 0,\ h(2) \to 1,\ h(3) \to 3,\ h(4) \to 2$ ***A1***
the groups are isomorphic ***AG***

$k(1) \to 0,\ k(2) \to 3,\ k(3) \to 1,\ k(4) \to 2$ ***A1***
is also a bijection

[7 marks]

Sub-total [16 marks]

continued …

Question 2 continued

Part B

(a) $h \otimes k \in$ group $\Rightarrow h \otimes k = e$ or h or k ***M1***

$h \otimes k = h \Rightarrow h \otimes k = h \otimes e \Rightarrow k = e$

but $k \neq e$ so $h \otimes k \neq h$ ***A1***

similarly $h \otimes k = k \Rightarrow h \otimes k = e \otimes k \Rightarrow h = e$

but $h \neq e$ so $h \otimes k \neq k$ ***A1***

hence $h \otimes k = e$ ***AG***

[3 marks]

(b) if cyclic then the group is $\{e, h, h^2\}$ ***R1***

$h^2 = e$ or h or k ***M1***

$h^2 = e \Rightarrow h \otimes h = h \otimes k$

$\Rightarrow h = k$

but $h \neq k$ so $h^2 \neq e$ ***A1***

$h^2 = h \Rightarrow h \otimes h = h \otimes e \Rightarrow h = e$

but $h \neq e$ so $h^2 \neq h$

so $h^2 = k$ ***A1***

also $h^3 = h \otimes k = e$ ***A1***

hence the group is cyclic ***AG***

Note: An alternative proof is possible based on order of elements and Lagrange.

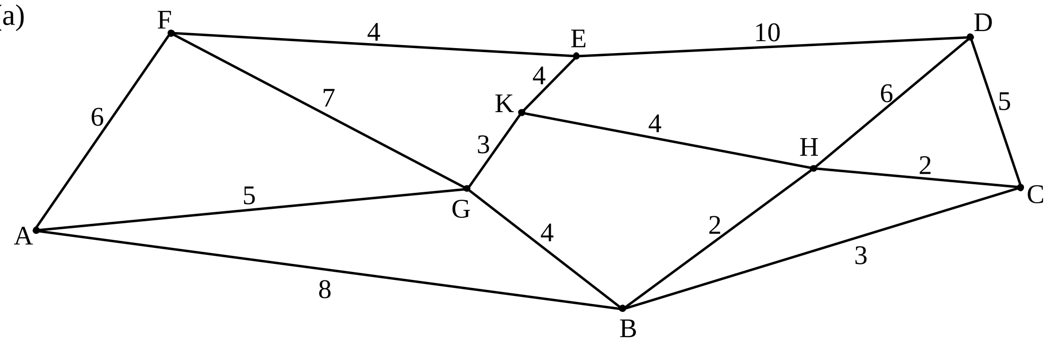
[5 marks]

Sub-total [8 marks]

Total [24 marks]

3. **Part A**

(a)



Start at an edge with weight 2, say BH, add other edges of weight 2 such that a cycle is not formed. Continue to add edges of increasing weight until all vertices have been included. ***M1***

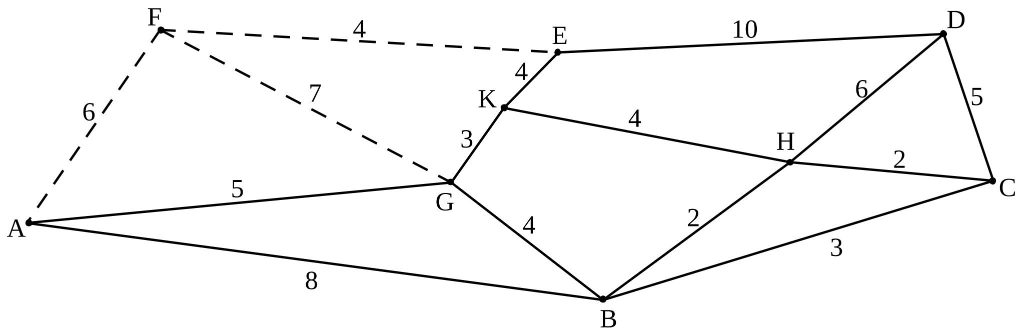
thus the minimum spanning tree is
$\text{BH}+\text{HC}+\text{GK}+\text{KH}+\text{KE}+\text{EF}+\text{GA}+\text{CD}$ ***A3***
total weight $=2+2+3+4+4+4+5+5=29$ ***A1***

Note: GB may replace KH and other orders are possible.

[5 marks]

(b) (i) upper bound $=2\times$ weight of minimum spanning tree ***M1***
$=58$ ***A1***

(ii) deleting vertex F ***M1***



the minimum spanning tree is
$\text{BH}+\text{HC}+\text{GK}+\text{KE}+\text{KH}+\text{GA}+\text{CD}$ ***A2***
total weight $=2+2+3+4+4+5+5=25$ ***A1***

adding the two edges of least weight from F ***M1***
lower bound $=25+4+6=35$ ***A1***

Note: Alternative solutions may be given by deleting a different vertex.

[8 marks]

Sub-total [13 marks]

continued …

Question 3 continued

Part B

(a) **EITHER**

$3 \mid m \Rightarrow m \equiv 0 (\text{mod } 3)$ ***(R1)***

if this is false then $m \equiv 1$ or $2 (\text{mod } 3)$ and $m^2 \equiv 1$ or $4 (\text{mod } 3)$ ***R1A1***

since $4 \equiv 1 (\text{mod } 3)$ then $m^2 \equiv 1 (\text{mod } 3)$ ***A1***

similarly $n^2 \equiv 1 (\text{mod } 3)$ ***A1***

hence $m^2 + n^2 \equiv 2 (\text{mod } 3)$

but $m^2 + n^2 \equiv 0 (\text{mod } 3)$ ***(R1)***

this is a contradiction so $3 \mid m$ and $3 \mid n$ ***R1AG***

[7 marks]

OR

$m \equiv 0, 1$ or $2 (\text{mod } 3)$ and $n \equiv 0, 1$ or $2 (\text{mod } 3)$ ***M1R1***

$\Rightarrow m^2 \equiv 0$ or $1 (\text{mod } 3)$ and $n^2 \equiv 0$ or $1 (\text{mod } 3)$ ***A1A1***

so $m^2 + n^2 \equiv 0, 1, 2 (\text{mod } 3)$ ***A1***

but $3 \mid m^2 + n^2$, so $m^2 + n^2 \equiv 0 (\text{mod } 3)$ ***R1***

$m \equiv 0 (\text{mod } 3)$ and $n \equiv 0 (\text{mod } 3)$ ***R1***

$\Rightarrow 3 \mid m$ and $3 \mid n$ ***AG***

[7 marks]

(b) suppose $\sqrt{2} = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and a and b are coprime ***M1***

then

$2b^2 = a^2$ ***A1***

$a^2 + b^2 = 3b^2$ ***A1***

$3b^2 \equiv 0 (\text{mod } 3)$ ***A1***

but by (a) a and b have a common factor so $\sqrt{2} \neq \frac{a}{b}$ ***R1***

$\Rightarrow \sqrt{2}$ is irrational ***AG***

[5 marks]

Sub-total [12 marks]

Total [25 marks]

4. **Part A**

(a) the general term is $\dfrac{(x+2)^n}{3^n n}$ ***A1***

[1 mark]

(b) $\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}}{a_n}\right)=\lim\limits_{n\to\infty}\left[\dfrac{(x+2)^{n+1}}{3^{n+1}(n+1)}\times\dfrac{3^n n}{(x+2)^n}\right]$ ***M1A1A1***

$=\lim\limits_{n\to\infty}\left[\dfrac{(x+2)n}{3(n+1)}\right]$ ***A1***

$=\dfrac{(x+2)}{3}$ since $\lim\limits_{n\to\infty}\left[\dfrac{n}{n+1}\right]=1$ ***A1R1***

the series is convergent if $\left|\dfrac{(x+2)}{3}\right|<1$ ***R1***

then $-3<x+2<3\Rightarrow -5<x<1$ ***A1***

if $x=-5$, series is $1-1+\dfrac{1}{2}-\dfrac{1}{3}+\ldots+\dfrac{(-1)^n}{n}+\ldots$ which converges ***M1A1***

if $x=1$, series is $1+1+\dfrac{1}{2}+\dfrac{1}{3}+\ldots+\dfrac{1}{n}+\ldots$ which diverges ***M1A1***

the interval of convergence is $-5\le x<1$ ***A1***

[13 marks]

Sub-total [14 marks]

Part B

$(u+3v^3)\dfrac{\mathrm{d}v}{\mathrm{d}u}=2v$

$\dfrac{\mathrm{d}u}{\mathrm{d}v}=\dfrac{(u+3v^3)}{2v}=\dfrac{u}{2v}+\dfrac{3v^2}{2}$ ***M1A1***

$\dfrac{\mathrm{d}u}{\mathrm{d}v}-\dfrac{u}{2v}=\dfrac{3v^2}{2}$ ***A1***

IF is $\mathrm{e}^{\int -\frac{1}{2v}\mathrm{d}v}=\mathrm{e}^{-\frac{1}{2}\ln v}$ ***M1***

$=v^{-\frac{1}{2}}$ ***A1***

$\dfrac{u}{\sqrt{v}}=\displaystyle\int\dfrac{3v^{\frac{3}{2}}}{2}\mathrm{d}v$ ***M1***

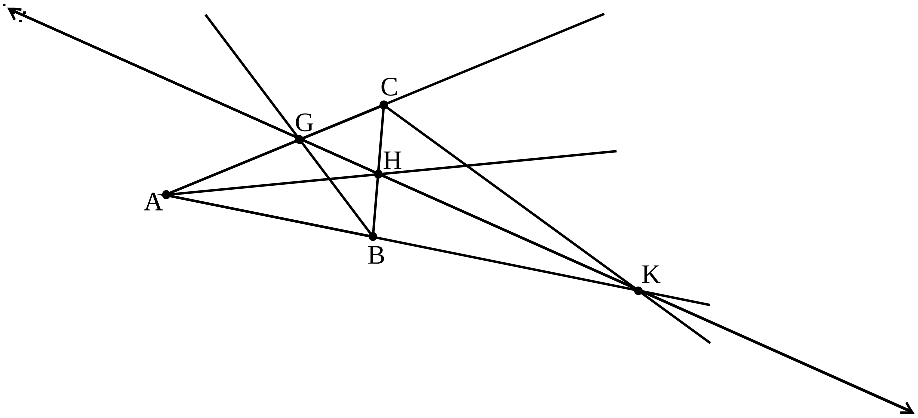
$=\dfrac{3}{5}v^{\frac{5}{2}}+c$ ***A1***

$u=\dfrac{3}{5}v^3+c\sqrt{v}$ ***A1***

Sub-total [8 marks]

Total [22 marks]

5. Part A



triangle ABC has interior angle bisectors AH, BG and exterior angle bisector CK
using the angle bisector theorem, ***M1***

$$\frac{CH}{HB}=\frac{CA}{AB};\ \frac{AG}{GC}=\frac{AB}{CB};\ \frac{BK}{AK}=\frac{CB}{CA}$$ ***A2***

hence, $\frac{CH}{HB}\times\frac{AG}{GC}\times\frac{BK}{AK}=\frac{CA}{AB}\times\frac{AB}{CB}\times\frac{CB}{CA}=1$ ***M1A1***

but $\frac{BK}{AK}=-\frac{BK}{KA}$ ***R1***

so, $\frac{CH}{HB}\times\frac{AG}{GC}\times\frac{BK}{AK}=-1$ ***A1***

hence, by converse Menelaus' theorem, G, H and K are collinear ***R1***

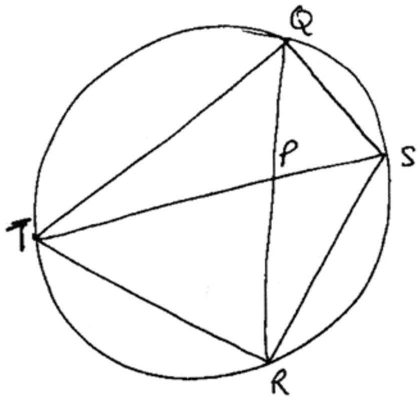
Sub-total [8 marks]

continued …

Question 5 continued

Part B

(a) (i)



$\text{ST} \bullet \text{QR} = \text{SQ} \bullet \text{RT} + \text{SR} \bullet \text{QT}$ ***M1A1***

but $\text{QT} = \text{QR} = \text{RT}$ ***R1***

hence $\text{ST} = \text{SQ} + \text{SR}$ ***AG***

(ii) $\angle\text{STR} = \angle\text{SQR}$ ***R1***

$\angle\text{QST} = \angle\text{QRT} = 60^\circ$ ***A1***

$\angle\text{RST} = \angle\text{RQT} = 60^\circ$ ***A1***

hence $\angle\text{QST} = \angle\text{RST}$ and $\Delta\text{RST} \sim \Delta\text{PSQ}$ ***R1***

(iii) $\dfrac{\text{ST}}{\text{SQ}} = \dfrac{\text{SR}}{\text{SP}} \rightarrow \text{ST} \times \text{SP} = \text{SR} \times \text{SQ}$ ***M1A1***

but $\text{ST} = (\text{SQ} + \text{SR})$

$(\text{SQ} + \text{SR}) \times \text{SP} = \text{SR} \times \text{SQ}$ ***A1***

$\text{SQ} \times \text{SP} + \text{SR} \times \text{SP} = \text{SR} \times \text{SQ}$

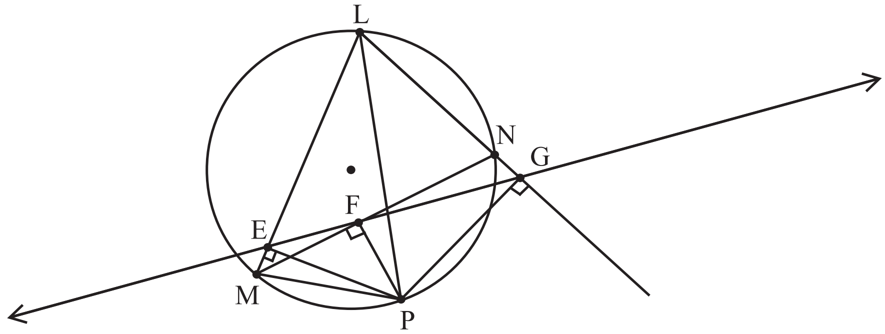
hence $\dfrac{1}{\text{SP}} = \dfrac{1}{\text{SQ}} + \dfrac{1}{\text{SR}}$ ***AG***

[10 marks]

continued …

Question 5 Part B continued

(b)



since $m\angle\text{PEM} \cong m\angle\text{PFM} \cong 90^\circ$
then quadrilateral PFEM is cyclic ***M1A1***
so $m\angle\text{PME} \cong m\angle\text{PFG}$ ***R1***
since $m\angle\text{PGL} \cong m\angle\text{PEL} \cong 90^\circ$
then quadrilateral PGLE is cyclic
so $m\angle\text{PGE} \cong m\angle\text{PLE}$ ***R1A1***
now E, F and G are collinear since they are on the Simson line of ΔLMN ***R1***
hence $\Delta\text{PFG} \sim \Delta\text{PML}$ ***A1***
so $\dfrac{\text{PL}}{\text{PM}} = \dfrac{\text{PG}}{\text{PF}} \Rightarrow \text{PL} \times \text{PF} = \text{PM} \times \text{PG}$ ***R1AG***

[8 marks]

Sub-total [18 marks]

Total [26 marks]