International Baccalaureate®
Baccalauréat International
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MARKSCHEME

May 2008

FURTHER MATHEMATICS

Standard Level

Paper 2

13 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**: often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) are often dependent on the preceding ***M*** mark.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). Apply a **MR** penalty of 1 mark to that question. Award the marks as usual and then write –1(**MR**) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled **(d)** and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates may not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

  Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

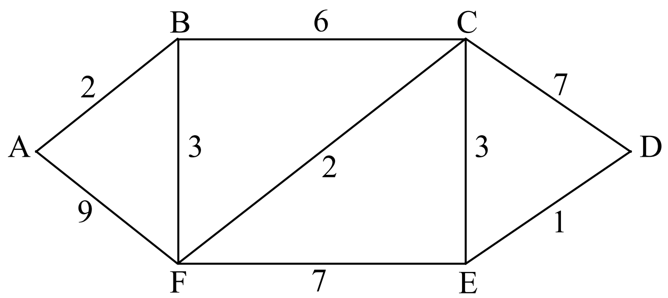
Candidates should be penalized ***once only IN THE PAPER*** *for an accuracy error* ***(AP)****. Award the marks as usual then write* ***(AP)*** *against the answer. On the* ***front*** *cover write –1****(AP)****. Deduct 1 mark from the total for the paper, not the question*.

- If a final correct answer is incorrectly rounded, apply the ***AP***.
- If the level of accuracy is not specified in the question, apply the ***AP*** for correct answers not given to three significant figures.

If there is no working shown, and answers are given to the correct two significant figures, apply the ***AP***. However, do **not** accept answers to one significant figure without working.

1. Part A

(a)



A2

[2 marks]

Note: The weights are not required for this ***A2***.

(b)

Iteration	Vertices	Labels
First	A	A (0) B (2) C (-) D (-) E (-) F (9)
Second	A, B	A (0) B (2) C (8) D (-) E (-) F (5)
Third	A, B, F	A (0) B (2) C (7) D (-) E (12) F (5)
Fourth	A, B, F, C	A (0) B (2) C (7) D (14) E (10) F (5)
Fifth	A, B, F, C, E	A (0) B (2) C (7) D (11) E (10) F (5)

M1A1A1A1A1A1

Shortest path is ABFCED ***A1***

Length $=11$ ***A1***

[8 marks]

Sub-total [10 marks]

Part B

(a) Multiply through by a^{p-2}.

$a^{p-1}x \equiv a^{p-2}b(\text{mod } p)$ ***M1A1***

Since, by Fermat's little theorem, $a^{p-1} \equiv 1(\text{mod } p)$, ***R1***

$x \equiv a^{p-2}b(\text{mod } p)$ ***AG***

[3 marks]

(b) Using the above result,

$x \equiv 3^3 \times 4(\text{mod } 5) \equiv 3(\text{mod } 5)$ ***M1A1***

$= 3, 8, 13, 18, 23, \ldots$ ***(A1)***

and $x \equiv 5^5 \times 6(\text{mod } 7) \equiv 4(\text{mod } 7)$ ***M1A1***

$= 4, 11, 18, 25, \ldots$ ***(A1)***

The general solution is

$x = 18 + 35n$ ***M1***

i.e. $x \equiv 18(\text{mod } 35)$ ***A1***

[8 marks]

Sub-total [11 marks]

Total [21 marks]

2. Part A

(a) $f'(x) = \dfrac{\cos x}{1+\sin x}$ ***M1A1***

$f''(x) = \dfrac{-\sin x(1+\sin x) - \cos^2 x}{(1+\sin x)^2}$ ***M1***

$= \dfrac{-\sin x - 1}{(1+\sin x)^2}$ ***A1***

$= \dfrac{-1}{1+\sin x}$ ***AG***

[4 marks]

(b) $f'''(x) = \dfrac{\cos x}{(1+\sin x)^2}$ ***A1***

$f^{iv}(x) = \dfrac{-\sin x(1+\sin x)^2 - 2(1+\sin x)\cos^2 x}{(1+\sin x)^4}$ ***A1***

$f(0) = 0,\ f'(0) = 1,\ f''(0) = -1,\ f'''(0) = 1,\ f^{iv}(0) = -2$ ***(A2)***

Note: Award ***A1*** for 2 errors and ***A0*** for more than 2 errors.

$\ln(1+\sin x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{6} - \dfrac{x^4}{12} + \ldots$ ***M1A1***

[6 marks]

(c) $\ln(1-\sin x) = \ln\big(1+\sin(-x)\big) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{6} - \dfrac{x^4}{12} + \ldots$ ***M1A1***

[2 marks]

(d) Adding, ***M1***

$\ln(1-\sin^2 x) = \ln\cos^2 x$ ***A1***

$= -x^2 - \dfrac{x^4}{6} + \ldots$ ***A1***

$\ln\cos x = -\dfrac{x^2}{2} - \dfrac{x^4}{12} + \ldots$ ***A1***

$\ln\sec x = \dfrac{x^2}{2} + \dfrac{x^4}{12} + \ldots$ ***AG***

[4 marks]

(e) $\dfrac{\ln\sec x}{x\sqrt{x}} = \dfrac{\sqrt{x}}{2} + \dfrac{x^2\sqrt{x}}{12} + \ldots$ ***M1***

Limit $= 0$ ***A1***

[2 marks]

Sub-total [18 marks]

continued …

Question 2 continued

Part B

(a) Interval width $= 26.1 - 22.7 = 3.4$

So $3.4 = 2z \times \dfrac{1.6}{\sqrt{5}}$ ***M1A1***

$z = 2.375\ldots$ ***A1***

Probability $= 0.9912$ ***A1***

Confidence level $= 2 \times 0.4912 = 98.2\%$ ***A1***

[5 marks]

(b) z-value $= 1.96$ ***A1***

We require

$$2 \times \frac{1.96 \times 1.6}{\sqrt{n}} < 2$$ ***M1A1***

Whence $n > 9.83$ ***A1***

So we need $n = 10$ ***A1***

Note: Accept $=$ signs throughout.

[5 marks]

Sub-total [10 marks]

Total [28 marks]

3. (a) (i) Because the triangles AGF and BHF are similar. ***R1***

(ii) It follows (by cyclic rotation or considering similar triangles) that

$$\frac{\text{BD}}{\text{DC}} = \frac{\text{BH}}{\text{IC}}$$ ***A1***

and $$\frac{\text{CE}}{\text{EA}} = \frac{\text{CI}}{\text{GA}}$$ ***A1***

Multiplying these three results gives Menelaus' Theorem, *i.e.*

$$\frac{\text{AF}}{\text{FB}} \times \frac{\text{BD}}{\text{DC}} \times \frac{\text{CE}}{\text{EA}} = \frac{\text{AG}}{\text{HB}} \times \frac{\text{BH}}{\text{IC}} \times \frac{\text{CI}}{\text{GA}}$$ ***M1A1***

$$= \frac{\text{AG}}{\text{GA}} \times \frac{\text{BH}}{\text{HB}} \times \frac{\text{CI}}{\text{IC}} = -1$$ ***M1A1***

(iii) The converse states that if D, E, F are points on the sides (BC), (CA), (AB) of a triangle such that

$$\frac{\text{AF}}{\text{FB}} \times \frac{\text{BD}}{\text{DC}} \times \frac{\text{CE}}{\text{EA}} = -1$$

then D, E, F are collinear. ***A1***

To prove this result, let D, E, F' be collinear points on the three sides so that, using the above theorem, ***M1***

$$\frac{\text{AF}'}{\text{F}'\text{B}} \times \frac{\text{BD}}{\text{DC}} \times \frac{\text{CE}}{\text{EA}} = -1$$ ***A1***

Since $\frac{\text{AF}}{\text{FB}} \times \frac{\text{BD}}{\text{DC}} \times \frac{\text{CE}}{\text{EA}} = -1$ ***M1***

$$\frac{\text{AF}'}{\text{F}'\text{B}} = \frac{\text{AF}}{\text{FB}}$$ ***A1***

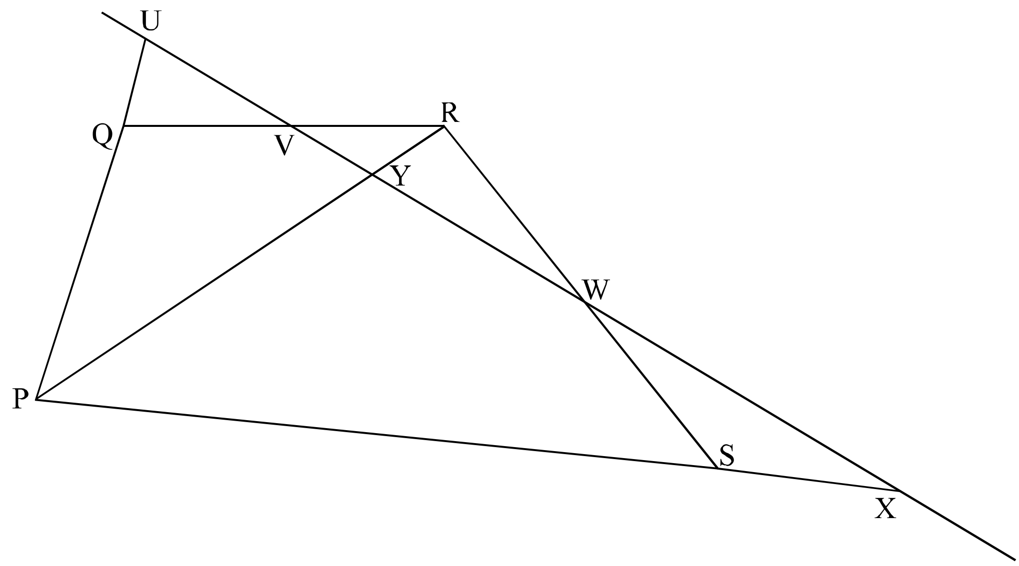
and $\text{F} = \text{F}'$ which proves the converse. ***R1***

[13 marks]

continued ...

Question 3 continued

(b)



Draw the diagonal PR and let it cut the line at the point Y. ***M1***
Apply Menelaus' Theorem to the triangle PQR. Then,

$$\frac{PU}{UQ}\times\frac{QV}{VR}\times\frac{RY}{YP}=-1$$ ***M1A1***

Now apply the theorem to triangle PRS.

$$\frac{PY}{YR}\times\frac{RW}{WS}\times\frac{SX}{XP}=-1$$ ***A1***

$$\frac{PU}{UQ}\times\frac{QV}{VR}\times\frac{RY}{YP}\times\frac{PY}{YR}\times\frac{RW}{WS}\times\frac{SX}{XP}=-1\times-1$$ ***M1***

$$\Rightarrow\frac{PU}{UQ}\times\frac{QV}{VR}\times\frac{RW}{WS}\times\frac{SX}{XP}\times\frac{PY}{YP}\times\frac{RY}{YR}=1$$ ***A1***

$$\Rightarrow\frac{PU}{UQ}\times\frac{QV}{VR}\times\frac{RW}{WS}\times\frac{SX}{XP}\times(-1)\times(-1)=1$$ ***(M1)***

$$\Rightarrow\frac{PU}{UQ}\times\frac{QV}{VR}\times\frac{RW}{WS}\times\frac{SX}{XP}=1$$ ***AG***

[7 marks]

Total [20 marks]

4. Part A

(a) (i) Since $a^2 - a^2 = 0$ is divisible by n, it follows that aR_1a so R_1 is reflexive. ***A1***

$aR_1b \Rightarrow a^2 - b^2$ divisible by $n \Rightarrow b^2 - a^2$ divisible by $n \Rightarrow bR_1a$ so symmetric. ***A1***

aR_1b and $bR_1c \Rightarrow a^2 - b^2 = pn$ and $b^2 - c^2 = qn$ ***A1***

$(a^2 - b^2) + (b^2 - c^2) = pn + qn$ ***M1***

so $a^2 - c^2 = (p+q)n \Rightarrow aR_1c$ ***A1***

Therefore R_1 is transitive.

It follows that R_1 is an equivalence relation. ***AG***

(ii) When $n = 8$, the equivalence classes are

$\{1, 3, 5, 7, 9, \ldots\}$, *i.e.* the odd integers ***A2***

$\{2, 6, 10, 14, \ldots\}$ ***A2***

and $\{4, 8, 12, 16, \ldots\}$ ***A2***

Note: If finite sets are shown award ***A1A1A1***.

[11 marks]

(b) Attempt to find a counter example. ***(M1)***

We note that $1R_2 6$ and $6R_2 11$ but 1 not $R_2 11$. ***A2***

Note: Accept any valid counter example.

The relation is not transitive. ***AG***

[3 marks]

Sub-total [14 marks]

Part B

Associativity follows since G is associative. ***A1***

Closure: Let $x, y \in H$ so $ax = xa, ay = ya$ for $a \in G$ ***M1***

Consider $axy = xay = xya \Rightarrow xy \in H$ ***M1A1***

The identity $e \in H$ since $ae = ea$ for $a \in G$ ***A2***

Inverse: Let $x \in H$ so $ax = xa$ for $a \in G$

Then

$x^{-1}a = x^{-1}axx^{-1}$ ***M1A1***

$= x^{-1}xax^{-1}$ ***M1***

$= ax^{-1}$ ***A1***

so $\Rightarrow x^{-1} \in H$ ***A1***

The four group axioms are satisfied so H is a subgroup. ***R1***

Sub-total [12 marks]

Total [26 marks]

5. (a) $\mathrm{P}(Z=n)=\sum_{k=0}^{n}\mathrm{e}^{-\lambda}\times\frac{\lambda^k}{k!}\times\mathrm{e}^{-\mu}\times\frac{\mu^{n-k}}{(n-k)!}$ ***M1A1***

$=\frac{\mathrm{e}^{-(\mu+\lambda)}}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\lambda^k\mu^{n-k}$ ***M1A1***

$=\frac{\mathrm{e}^{-(\mu+\lambda)}}{n!}(\lambda+\mu)^n$ ***A1***

This shows that Z is Poisson distributed with mean $(\lambda+\mu)$. ***R1***

[6 marks]

(b) The result is (trivially) true for $n=1$. ***A1***

Assuming it to be true for $n=k$, *i.e.* $\sum_{r=1}^{k}U_r\sim\mathrm{Po}(km)$ ***M1***

Consider $\sum_{r=1}^{k+1}U_r=\sum_{r=1}^{k}U_r+U_{k+1}$ ***M1A1***

which, using (a) is $\mathrm{Po}(km+m)$ *i.e.* $\mathrm{Po}([k+1]m)$ ***A1***

Hence proved by induction since true for $n=k\Rightarrow$ true for $n=k+1$
and we have shown true for $n=1$. ***R1***

[6 marks]

Total [12 marks]

6. (a) $S_{2n} = S_n + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$ ***M1***

$> S_n + \frac{1}{2n} + \frac{1}{2n} + \ldots + \frac{1}{2n}$ ***M1A1***

$= S_n + \frac{1}{2}$ ***AG***

[3 marks]

(b) Replacing n by $2n$,

$S_{4n} > S_{2n} + \frac{1}{2}$ ***M1A1***

$> S_n + 1$ ***A1***

Continuing this process,

$S_{8n} > S_n + \frac{3}{2}$ ***(A1)***

In general,

$S_{2^m n} > S_n + \frac{m}{2}$ ***M1A1***

Putting $n = 2$ ***M1***

$S_{2^{m+1}} > S_2 + \frac{m}{2}$ ***AG***

[7 marks]

(c) Consider the (large) number N. ***M1***

Then, $S_{2^{m+1}} > N$ if $S_2 + \frac{m}{2} > N$ ***A1***

i.e. if $m > 2(N - S_2)$ ***A1***

This establishes the divergence. ***AG***

[3 marks]

Total [13 marks]