22087102

International Baccalaureate®
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FURTHER MATHEMATICS
STANDARD LEVEL
PAPER 2

Tuesday 20 May 2008 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. All students should therefore be advised to show their working.

1. *[Total mark: 21]*

Part A *[Maximum mark: 10]*

The graph G has the following cost adjacency matrix.

	A	B	C	D	E	F
A	–	2	–	–	–	9
B	2	–	6	–	–	3
C	–	6	–	7	3	2
D	–	–	7	–	1	–
E	–	–	3	1	–	7
F	9	3	2	–	7	–

(a) Draw G in planar form. *[2 marks]*

(b) Use Dijkstra's Algorithm to find the shortest path between the vertices A and D. Show all the steps in the algorithm and state the length of the shortest path. *[8 marks]*

Part B *[Maximum mark: 11]*

(a) Given that $ax \equiv b \pmod{p}$ where $a, b, p, x \in \mathbb{Z}^+$, p is prime and a is not a multiple of p, use Fermat's little theorem to show that

$$x \equiv a^{p-2} b \pmod{p}.$$ *[3 marks]*

(b) Hence solve the simultaneous linear congruences

$$3x \equiv 4 \pmod{5}$$
$$5x \equiv 6 \pmod{7}$$

giving your answer in the form $x \equiv c \pmod{d}$. *[8 marks]*

2. *[Total mark: 28]*

Part A *[Maximum mark: 18]*

The function f is defined by $f(x) = \ln(1+\sin x)$.

(a) Show that $f''(x) = \dfrac{-1}{1+\sin x}$. *[4 marks]*

(b) Determine the Maclaurin series for $f(x)$ as far as the term in x^4. *[6 marks]*

(c) Deduce the Maclaurin series for $\ln(1-\sin x)$ as far as the term in x^4. *[2 marks]*

(d) By combining your two series, show that $\ln \sec x = \dfrac{x^2}{2} + \dfrac{x^4}{12} + \ldots$. *[4 marks]*

(e) Hence, or otherwise, find $\lim\limits_{x \to 0} \dfrac{\ln \sec x}{x\sqrt{x}}$. *[2 marks]*

Part B *[Maximum mark: 10]*

When a scientist measures the concentration μ of a solution, the measurement obtained may be assumed to be a normally distributed random variable with mean μ and standard deviation 1.6.

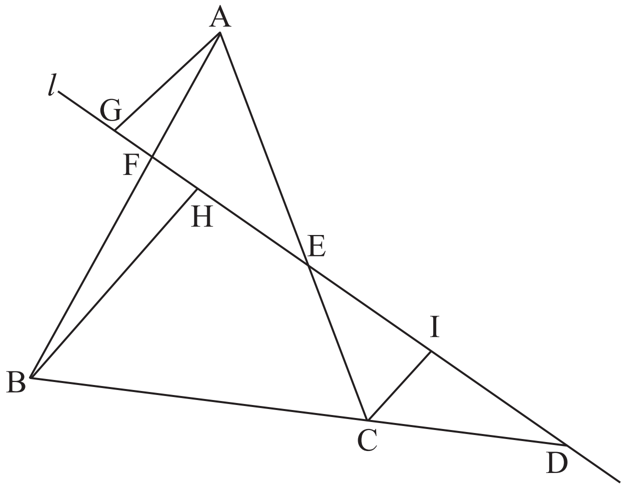
(a) He makes 5 independent measurements of the concentration of a particular solution and correctly calculates the following confidence interval for μ.

$$[22.7, 26.1]$$

Determine the confidence level of this interval. *[5 marks]*

(b) He is now given a different solution and is asked to determine a 95 % confidence interval for its concentration. The confidence interval is required to have a width less than 2. Find the minimum number of independent measurements required. *[5 marks]*

3. *[Maximum mark: 20]*



(a) The diagram shows the line l meeting the sides of the triangle ABC at the points D, E and F. The perpendiculars to l from A, B and C meet l at G, H and I.

(i) State why $\frac{\text{AF}}{\text{FB}} = \frac{\text{AG}}{\text{HB}}$.

(ii) Hence prove Menelaus' theorem for the triangle ABC.

(iii) State and prove the converse of Menelaus' theorem. *[13 marks]*

(b) A straight line meets the sides (PQ), (QR), (RS), (SP) of a quadrilateral PQRS at the points U, V, W, X respectively. Use Menelaus' theorem to show that

$$\frac{\text{PU}}{\text{UQ}} \times \frac{\text{QV}}{\text{VR}} \times \frac{\text{RW}}{\text{WS}} \times \frac{\text{SX}}{\text{XP}} = 1.$$

[7 marks]

4. *[Total mark: 26]*

Part A *[Maximum mark: 14]*

(a) The relation R_1 is defined for $a, b \in \mathbb{Z}^+$ by aR_1b if and only if $n \mid (a^2 - b^2)$ where n is a fixed positive integer.

(i) Show that R_1 is an equivalence relation.

(ii) Determine the equivalence classes when $n = 8$. *[11 marks]*

(b) The relation R_2 is defined for $a, b \in \mathbb{Z}^+$ by aR_2b if and only if $(4 + |a - b|)$ is the square of a positive integer. Show that R_2 is not transitive. *[3 marks]*

Part B *[Maximum mark: 12]*

Consider the group $\{G, *\}$ and let H be a subset of G defined by

$$H = \{x \in G \text{ such that } x * a = a * x \text{ for all } a \in G\}.$$

Show that $\{H, *\}$ is a subgroup of $\{G, *\}$.

5. *[Maximum mark: 12]*

(a) The independent random variables X and Y have Poisson distributions and $Z = X + Y$. The means of X and Y are λ and μ respectively. By using the identity

$$\mathrm{P}(Z = n) = \sum_{k=0}^{n} \mathrm{P}(X = k)\,\mathrm{P}(Y = n - k)$$

show that Z has a Poisson distribution with mean $(\lambda + \mu)$. *[6 marks]*

(b) Given that $U_1, U_2, U_3, \ldots$ are independent Poisson random variables each having mean m, use mathematical induction together with the result in (a) to show that $\sum_{r=1}^{n} U_r$ has a Poisson distribution with mean nm. *[6 marks]*

6. *[Maximum mark: 13]*

Let $S_n = \sum_{k=1}^{n} \frac{1}{k}$.

(a) Show that, for $n \geq 2$, $S_{2n} > S_n + \frac{1}{2}$. *[3 marks]*

(b) Deduce that $S_{2^{m+1}} > S_2 + \frac{m}{2}$. *[7 marks]*

(c) Hence show that the sequence $\{S_n\}$ is divergent. *[3 marks]*