

MARKSCHEME

MAY 2006

FURTHER MATHEMATICS

Standard Level

Paper 2

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**; working must be seen.
- (M)** Marks awarded for **Method**; may be implied by **correct** subsequent working.
- A** Marks awarded for an **Answer** or for **Accuracy**: often dependent on preceding **M** marks.
- (A)** Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.
- R** Marks awarded for clear **Reasoning**.
- N** Marks awarded for **correct** answers if **no** working shown (or working which gains no other marks).
- AG** Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations **MI**, **AI**, etc.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award **M0** followed by **AI**, as **A** mark(s) are often dependent on the preceding **M** mark.
- Where **M** and **A** marks are noted on the same line, e.g. **MIAI**, this usually means **MI** for an **attempt** to use an appropriate method (e.g. substitution into a formula) and **AI** for using the **correct** values.
- Where the markscheme specifies **(M2)**, **N3**, etc, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 N marks

*Award N marks for **correct** answers where there is **no** working, (or working which gains no other marks).*

- Do **not** award a mixture of **N** and other marks.
- There may be fewer **N** marks available than the total of **M**, **A** and **R** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.
- For consistency within the markscheme, **N** marks are noted for every part, even when these match the mark breakdown. In these cases, the marks may be recorded in either form e.g. **A2** or **N2**.

4 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks.
- If the error leads to an inappropriate value (e.g. $\sin \theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **A** marks can be awarded, but **M** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). Apply a **MR** penalty of 1 mark to that question. Award the marks as usual and then write $-1(\mathbf{MR})$ next to the total. Subtract 1 mark from the total for the question. A candidate should be penalised only once for a particular mis-read.*

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. $\sin \theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks (*d*)

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled (**d**) and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for part-questions are indicated by **EITHER . . . OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates may not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2 \sin(5x - 3)$, the markscheme gives:

$$f'(x) = (2 \cos(5x - 3)) 5 \quad (= 10 \cos(5x - 3)) \quad \mathbf{AI}$$

Award **AI** for $(2 \cos(5x - 3)) 5$, even if $10 \cos(5x - 3)$ is not seen.

10 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

- **Rounding errors:** only applies to final answers not to intermediate steps.
- **Level of accuracy:** when this is not specified in the question the general rule applies: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Candidates should be penalized **once only IN THE PAPER** for an accuracy error (**AP**). Award the marks as usual then write (**AP**) against the answer. On the **front** cover write – 1(**AP**). Deduct 1 mark from the total for the paper, not the question.

- If a final correct answer is incorrectly rounded, apply the **AP**.
- If the level of accuracy is not specified in the question, apply the **AP** for correct answers not given to three significant figures.

If there is no working shown, and answers are given to the correct two significant figures, apply the **AP**. However, do **not** accept answers to one significant figure without working.

11 Examples

Exemplar material is available under examiner training on examnet. Please refer to this material before you start marking, and when you have any queries. Please also feel free to contact your Team Leader if you need further advice.

1. Part A

- (a) (i) The number of hits is binomial with $n = 8$, $p = 0.6$. *(M1)*
 Probability = $\binom{8}{5} \times 0.6^5 \times 0.4^3 = 0.279$ *M1A1* *N2*
- (ii) We now use the negative binomial distribution. *(M1)*
 Probability = $\binom{7}{4} \times 0.6^5 \times 0.4^3 = 0.174$ *M1A1* *N2*
- [6 marks]*
- (b) Mean = $\frac{10}{0.6} = 16.7$ *M1A1* *N2*
 Variance = $\frac{10 \times 0.4}{0.6^2} = 11.111\dots \left(\frac{100}{9}\right)$ *M1(A1)*
 SD = $3.33 \left(\frac{10}{3}\right)$ *A1* *N3*
- [5 marks]*

Part B

- (a) $H_0 : p = 0.5$ $H_1 : p > 0.5$ *A1A1*
[2 marks]
- (b) (i) Critical (region). *A1*
- (ii) The significance level is the probability of rejecting H_0 when it is true. *A2*
 Significance level = 0.0592 *A2*
- [5 marks]*
- (c) $P(\text{Type II error}) = P(X \leq 10 | p = 0.6)$ *M1A1*
 $= 0.783$ *A2* *N2*
- [4 marks]*
- (d) (i) She commits a Type II error. *R2*
- (ii) She would conclude that the coin is fair when it is not. *R2*
- [4 marks]*

Total [26 marks]

2. Part A

(a)
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A2$$

[2 marks]

(b) Closure:
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c+a & d+b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G \quad M1A1$$

Identity:
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G \quad M1A1$$

Inverse: It follows from (a) that

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & -b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G \quad A2$$

Abelian: Consider

$$\begin{pmatrix} 1 & c & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+c & b+d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Comparing this with the result proving closure above and noting that $a+c=c+a$ and $b+d=d+c$ shows that the group is Abelian. A2

Associativity: this follows since matrix multiplication is associative. A1

We have established the 5 criteria for an Abelian group. R1

[10 marks]

(c) Consider the 1:1 correspondence $\phi \left(\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = (a, b) \quad (M1)$

Then

$$\phi \left(\left(\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \right) = \phi \begin{pmatrix} 1 & a+c & b+d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M1$$

$$= (a+c, b+d) \quad A1$$

$$= (a, c) + (b, d) \quad A1$$

$$= \phi \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \phi \begin{pmatrix} 1 & c & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A1$$

This shows that ϕ defines an isomorphism. R1

[6 marks]

continued...

Question 2 continued

Part B

- (a) Consider $(x \cdot y) \cdot (y^{-1} \cdot x^{-1}) = x \cdot (y \cdot y^{-1}) \cdot x^{-1}$ *A1*
 $= x \cdot e \cdot x^{-1}$ *A1*
 $= x \cdot x^{-1}$ *A1*
 $= e$ *A1*
- Therefore
 $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$ *AG* *N0*
[4 marks]
- (b) Reflexive: $a = e \cdot a \cdot e^{-1} \Rightarrow a R a$ *M1A1*
- Symmetric: $a R b \Rightarrow a = c \cdot b \cdot c^{-1} \Rightarrow b = c^{-1} \cdot a \cdot c$ *M1A1*
 $\Rightarrow b = c^{-1} \cdot a \cdot (c^{-1})^{-1} \Rightarrow b R a$ *A1*
- Transitive: $a R b$ and $b R c \Rightarrow a = d \cdot b \cdot d^{-1}$ and $b = f \cdot c \cdot f^{-1}$ *M1A1*
 $\Rightarrow a = (d \cdot f) \cdot c \cdot (d \cdot f)^{-1} \Rightarrow a R c$ *A1*
- Therefore R is an equivalence relation. *R1*
[9 marks]
- Total [31 marks]*

3. Part A

- (a) $110 = 75 + 35$ *A1*
 $75 = 2 \times 35 + 5$ *A1*
 $35 = 7 \times 5$ *A1*
 Therefore, $\text{gcd}(75, 110) = 5$. *A1*

[4 marks]

- (b) (i) The gcd (5) is a factor of the right hand side (45). *R1*

- (ii) Working backwards,
 $5 = 75 - 2 \times 35$ *M1*
 $= 75 - 2 \times (110 - 75)$ *A1*
 $= 3 \times 75 - 2 \times 110$ *A1*
 Multiplying by 9,
 $-18 \times 110 + 27 \times 75 = 45$ *A1*
 A solution is therefore $(-18, 27)$. *A1*

- (iii) The general solution is $(-18 + 15m, 27 - 22m)$. *A1*

[7 marks]

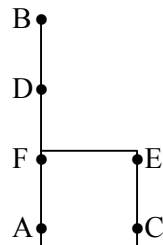
Part B

- (a) For this graph, $v = 6$ and $e = 15$. *R1*
 The condition for a planar graph is $e \leq 3v - 6$. *R1*
 This condition is not satisfied so the graph is not planar. *R2*

[4 marks]

- (b) Different notations may be used but the edges should be added in the following order. *(M1)*
 Using Prim's Algorithm,

- BD *A1*
 DF *A1*
 FA *A1*
 FE *A1*
 EC *A1*



A2

Total weight = 12 *A2*

[10 marks]

Total [25 marks]

4. Part A

Noting either that the error in an alternating series is less than the next term or using the Lagrange form of the error term in the Maclaurin series,

$$\text{Error} \leq \frac{|x^7|}{7!} \quad \text{M1A1}$$

We require

$$\frac{|x^7|}{7!} < 0.005 \quad \text{M1}$$

$$|x^7| < 25.2 \quad \text{(A1)}$$

$$|x| < 1.59 \text{ (Accept } |x| \leq 1.59) \quad \text{A1}$$

[5 marks]

Part B

We first note that

$$(n+1)^{-n} \leq 2^{-n} \text{ for } n \geq 1 \quad \text{M1A1}$$

$$\sum 2^{-n} \text{ is a geometric series with common ratio } 0.5 \quad \text{R2}$$

It is therefore convergent. R1

$$\sum (n+1)^{-n} \text{ is therefore convergent by the comparison test.} \quad \text{M1A1}$$

[7 marks]

Total [12 marks]

5. $y(0.1) = y(0) + 0.1(2 \times 1 + \sin(0)) \quad \text{M1A1}$

$$= 1.2 \quad \text{A1}$$

$$y(0.2) = 1.2 + 0.1(2 \times 1.2 + \sin(0.1)) \quad \text{M1A1}$$

$$= 1.449983342 \text{ (1.45)} \quad \text{A1}$$

$$y(0.3) = 1.449983342 + 0.1(2 \times 1.449983342 + \sin(0.2)) \quad \text{A1}$$

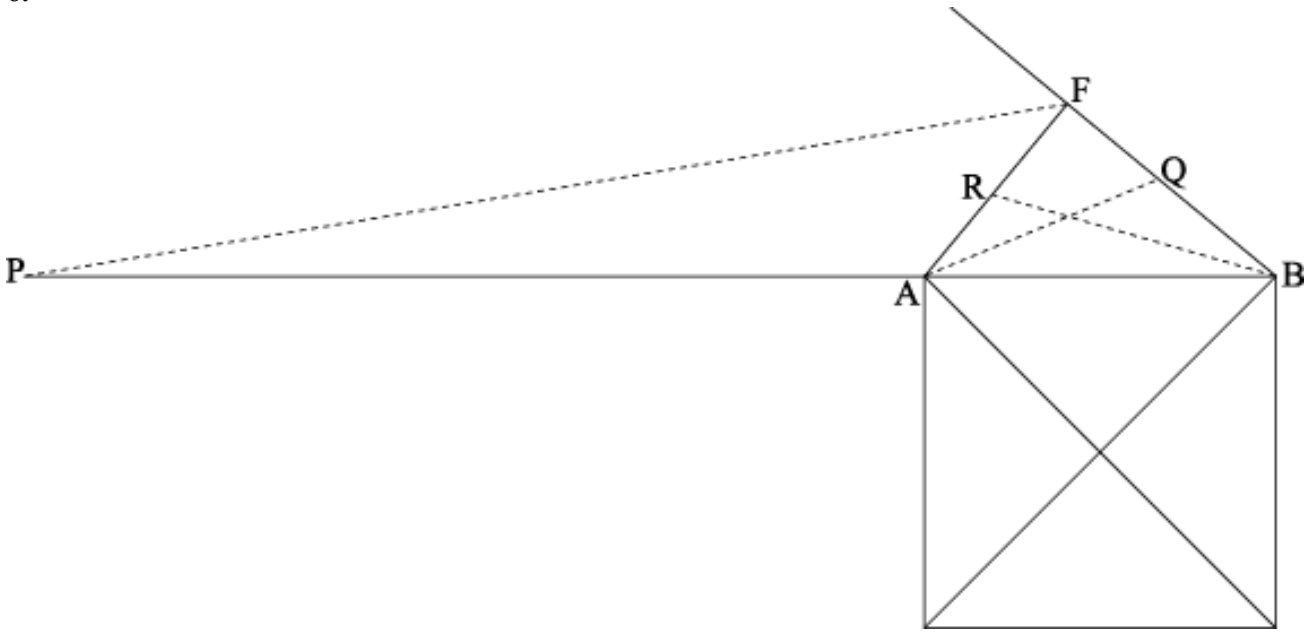
$$= 1.759846943 \text{ (1.76)} \quad \text{A1}$$

$$y(0.4) = 1.759846943 + 0.1(2 \times 1.759846943 + \sin(0.3)) \quad \text{A1}$$

$$= 2.14 \quad \text{A1} \quad \text{N0}$$

Total [10 marks]

6.



(a) $AB = \sqrt{a^2 + b^2}$ (A1)

$EB = AE = AB \times \frac{\sqrt{2}}{2} = \sqrt{a^2 + b^2} \times \frac{\sqrt{2}}{2}$ (M1)(A1)

Since both angles E and F are right angles, the quadrilateral AFBE is cyclic. R1

By Ptolemy's theorem:

$AF \times EB + FB \times AE = EF \times AB$, i.e. (M1)

$b \times \sqrt{a^2 + b^2} \times \frac{\sqrt{2}}{2} + a \times \sqrt{a^2 + b^2} \times \frac{\sqrt{2}}{2} = EF \times \sqrt{a^2 + b^2}$ A1

Since $\sqrt{a^2 + b^2} \neq 0$, then (M1)

$EF = \frac{\sqrt{2}}{2}(a + b)$ A1

[8 marks]

(b) Since (FP) is the external bisector of angle F , then **(M1)**

$$\frac{AP}{PB} = \frac{AF}{FB} = -\frac{b}{a} \quad \text{A1}$$

Similarly, (AR) is the bisector of angle A , then **(M1)**

$$\frac{BR}{RF} = \frac{BA}{AF} = \frac{\sqrt{a^2 + b^2}}{b} \quad \text{A1}$$

Also, (BQ) bisects angle B , then

$$\frac{FQ}{QA} = \frac{FB}{BA} = \frac{a}{\sqrt{a^2 + b^2}} \quad \text{A1}$$

$$\text{So, } \frac{BR}{RF} \times \frac{FQ}{QA} \times \frac{AP}{PB} = \frac{\sqrt{a^2 + b^2}}{b} \times \frac{a}{\sqrt{a^2 + b^2}} \times -\frac{b}{a} = -1 \quad \text{M1A1}$$

Therefore, by (the converse of) Menelaus' theorem, the points P, Q, and R are collinear.

R1

[8 marks]

Total [16 marks]
