MARKSCHEME

November 2005

FURTHER MATHEMATICS

Standard Level

Paper 2

Paper 2 Markscheme

Instructions to Examiners

1 Method of marking

(a) All marking must be done using a **red** pen.

(b) Marks should be noted on candidates' scripts as in the markscheme:

- show the breakdown of individual marks using the abbreviations ***(M1)***, ***(A2)*** etc.
- write down each part mark total, indicated on the markscheme (for example, ***[3 marks]***) – it is suggested that this be written at the end of each part, and underlined;
- write down and circle the total for each question at the end of the question.

2 Abbreviations

The markscheme may make use of the following abbreviations:

M Marks awarded for **Method**

A Marks awarded for an **Answer** or for **Accuracy**

G Marks awarded for correct solutions, generally obtained from a **Graphic Display Calculator**, irrespective of working shown

R Marks awarded for clear **Reasoning**

AG **Answer Given** in the question and consequently marks are **not** awarded

3 Follow Through (ft) Marks

Errors made at any step of a solution can affect all working that follows. To limit the severity of the penalty, **follow through (ft)** marks should be awarded. The procedures for awarding these marks require that all examiners:

(i) penalise an error when it **first occurs**;

(ii) **accept the incorrect answer** as the appropriate value or quantity to be used in all subsequent working;

(iii) award ***M*** marks for a correct method, and ***A*****(ft)** marks if the subsequent working contains no further errors.

Follow through procedures may be applied repeatedly throughout the same problem.

The following illustrates a use of the **follow through** procedure:

Markscheme		Candidate's Script	Marking	
$ 600 × 1.02	***M1***	Amount earned = $ 600 × 1.02	√	***M1***
= $ 612	***A1***	= $602	×	***A0***
$ (306 × 1.02) + (306 × 1.04)	***M1***	Amount = 301 × 1.02 + 301 × 1.04	√	***M1***
= $ 630.36	***A1***	= $ 620.06	√	***A1*(ft)**

Note that the candidate made an arithmetical error at line 2; the candidate used a correct method at lines 3, 4; the candidate's working at lines 3, 4 is correct.

However, if a question is transformed by an error into a **different, much simpler question** then:

(i) **fewer** marks should be awarded at the discretion of the Examiner;

(ii) marks awarded should be followed by "**(d)**" (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(iii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

4 Using the Markscheme

(a) This markscheme presents a particular way in which each question may be worked and how it should be marked. **Alternative methods** have not always been included. Thus, if an answer is wrong then the working must be carefully analysed in order that marks are awarded for a different method in a manner which is consistent with the markscheme.

In this case:

(i) a mark should be awarded followed by "**(d)**" (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(ii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

Where alternative methods for complete questions are included, they are indicated by **METHOD 1**, **METHOD 2**, *etc*. Other alternative solutions, including graphic display calculator alternative solutions are indicated by **OR**. For example:

Mean = 7906/134 ***(M1)***
= 59 ***(A1)***

OR

Mean = 59 ***(G2)***

(b) Unless the question specifies otherwise, accept **equivalent forms**. For example: $\frac{\sin\theta}{\cos\theta}$ for $\tan\theta$.

On the markscheme, these equivalent numerical or algebraic forms will generally be written in brackets after the required answer. Paper setters will indicate the required answer, by allocating full marks at that point. Further working should be ignored, even if it is incorrect. For example: if candidates are asked to factorize a quadratic expression, and they do so correctly, they are awarded full marks. If they then continue and find the roots of the corresponding equation, do not penalize, even if those roots are incorrect i.e. once the correct answer is seen, ignore further working.

(c) As this is an international examination, all **alternative forms of notation** should be accepted. For example: 1.7, $1 \cdot 7$, 1,7; different forms of vector notation such $\vec{u}$, $\overline{u}$, $\underline{u}$, $\tan^{-1} x$ for $\arctan x$.

5 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

There are two types of accuracy error. Candidates should be penalized **once only IN THE PAPER** for an accuracy error **(AP)**.

Award the marks as usual then write -1**(AP)** against the answer and also on the **front** cover.

Rounding errors: only applies to final answers not to intermediate steps.

Level of accuracy: when this is not specified in the question the general rule *unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures* applies.

- If a final correct answer is incorrectly rounded, apply the **AP**.

  OR

- If the level of accuracy is not specified in the question, apply the **AP** for answers not given to 3 significant figures. (Please note that this has changed from May 2003).

Incorrect answers are wrong, and the accuracy penalty should not be applied to incorrect answers.

Examples

A question leads to the answer 4.6789….
- 4.68 is the correct 3 s.f. answer.
- 4.7, 4.679 are to the wrong level of accuracy : both should be penalised the first time this type of error occurs.
- 4.67 is incorrectly rounded - penalise on the first occurrence.

Note: All these "incorrect" answers may be assumed to come from 4.6789..., even if that value is not seen, but previous correct working is shown. However, 4.60 is wrong, as is 4.5, 4.8, and these should be penalised as being incorrect answers, not as examples of accuracy errors.

6 Graphic Display Calculators

Many candidates will be obtaining solutions directly from their calculators, often without showing any working. They have been advised that they must use mathematical notation, not calculator commands when explaining what they are doing. Incorrect answers without working will receive no marks. However, if there is written evidence of using a graphic display calculator correctly, method marks may be awarded. Where possible, examples will be provided to guide examiners in awarding these method marks.

1. (i) (a) An appropriate model is the Poisson on the basis of the following assumptions: (Accept any three of the following conditions)

(i) Accidents occur independently,

(ii) Simultaneous accidents are not possible,

(iii) accidents happen randomly over time,

(iv) accidents occur uniformly (Mean number per time period is proportional to the period length) ***(A3)***

[3 marks]

(b) A Poisson distribution has one parameter λ that needs to be estimated: ***(AG)***

$\lambda \simeq \bar{x} = \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{325}{260} = 1.25$ ***(A1)***

Also as a check on the closeness of the data to a probable Poisson we can check the variance

$\text{variance} = \dfrac{\sum f_i x_i^2}{\sum f_i} - (\bar{x})^2 = \dfrac{735}{260} - 1.25^2 = 1.26$ ***(R1)***

Now, with $\lambda = 1.25$, we can calculate the expected probabilities and the corresponding frequencies using the Poisson probability density function and the fact that

$E_i = n \times \mathrm{P}(X = x_i)$. ***(M1)***

x_i	0	1	2	3	4	5	≥ 6
$\mathrm{P}(x_i)$	0.2865	0.3581	0.2238	0.0933	0.0291	0.0073	0.0019
E_i	74.5	93.1	58.2	24.2	7.6	1.9	0.5

(A1)

The test statistic χ^2 will be used with the provision that no expected frequency is less than 5. Hence the last three cells will have to be combined. ***(R1)***

x_i	0	1	2	3	≥ 4
O_i	77	90	55	30	8
E_i	74.5	93.1	58.2	24.2	10

(A1)

H_0 : number of weekly accidents ~ Po(1.25)

H_1 : number of weekly accidents is not ~ Po(1.25) ***(A1)***

Degrees of freedom, $\upsilon = 5 - 1 - 1 = 3$

$\chi^2 = \sum \dfrac{(O_i - E_i)^2}{E_i} = 2.153 < \chi^2_{3,\,0.95} = 7.815$. Hence there is no evidence to reject the hypothesis that the data follows a Poisson distribution with mean 1.25. ***(M1)(A1)(R1)***

[10 marks]

continued ...

Question 1 continued

(ii) (a) $H_0: \mu = 0$
$H_1: \mu > 0$ ***(A1)***

[1 mark]

(b) (i) We use the t-distribution, with 24 degrees of freedom ***(R1)***
The critical value is 1.71 ***(A1)***

t-value is $\dfrac{\bar{x}-0}{\frac{s}{\sqrt{25}}} = \dfrac{0.3724-0}{\frac{0.6724}{5}} = 2.77$ ***(M1)(A1)***

(ii) Since $2.77 > 1.71$, we reject H_0, and conclude that compiler II is faster than compiler I. ***(R1)(A1)***

[6 marks]

Total [20 marks]

2. (i) (a) The order of the group G must be a multiple of the order of subgroups S and T. The lowest common multiple of 3 and 4 is 12. ***(R1)(A1)***

[2 marks]

(b) If the order of group of is 12, we need 6 more elements, ***(R1)***
for the group to be commutative. Since the order of a^2 is 2
the order of a^2b and a^2b^2 will be 6, so the generators are ***(R1)***
ab, ab^2, a^3b and a^3b^2. ***(A4)***

[6 marks]

(ii) (a) Closure: $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix} \in H$. ***(M1)(A1)***

The identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in H$ ***(A1)***

Inverse: $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \in H$. ***(M1)(A1)***

[5 marks]

(b) Define $f : \mathbb{R} \to H$ by $f(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, for all $x \in \mathbb{R}$. ***(R1)***

This is a 1-1 function since

$f(x_1) = f(x_2) \Rightarrow \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix}$

and so $x_1 = x_2$, ***(M1)(A1)***

The function also preserves the group operation, i.e.

$f(x_1 + x_2) = \begin{pmatrix} 1 & x_1 + x_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} = f(x_1) \bullet f(x_1)$ ***(M1)(A1)***

[5 marks]

Total [18 marks]

3. (i) (a)

$$T=\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 0 \\ b & 1 & 0 & 2 & 0 \\ c & 1 & 2 & 0 & 1 \\ d & 0 & 0 & 1 & 0 \end{array}$$

(A2)

[2 marks]

(b) (i)

$$T^2=\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 2 & 2 & 2 & 1 \\ b & 2 & 5 & 1 & 2 \\ c & 2 & 1 & 6 & 0 \\ d & 1 & 2 & 0 & 1 \end{array}$$

(A1)

For $\boldsymbol{T}^2$ the entries represent the **number of** paths of length 2 that exist between the different vertices. *(A1)*

(ii)

$$T+T^2=\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 2 & 3 & 3 & 1 \\ b & 3 & 5 & 3 & 2 \\ c & 3 & 3 & 6 & 1 \\ d & 1 & 2 & 1 & 1 \end{array}$$

(A1)

For $\boldsymbol{T}+\boldsymbol{T}^2$ entries represent the **number of** paths of maximum length 2 that exist between the different vertices. *(A1)*

[4 marks]

(ii) (a) Since r is a repeated root of the *characteristic* polynomial, then $x^2-px-q=(x-r)^2 \Rightarrow p=2r$ and $q=-r^2$. *(A2)*

If $a_n = Ar^n + Bnr^n$ is a solution of the difference equation, then when substituted into the equation it must be a true statement:

$$pa_{n-1}+qa_{n-2}=p(Ar^{n-1}+B(n-1)r^{n-1})+q(Ar^{n-2}+B(n-2)r^{n-2})$$ *(M1)*

$$=2r(Ar^{n-1}+B(n-1)r^{n-1})-r^2(Ar^{n-2}+B(n-2)r^{n-2})$$ *(A1)*

$$=2Ar^n+2B(n-1)r^n-Ar^n-B(n-2)r^n$$

$$=Ar^n+Bnr^n$$

$$=a_n$$ *(M1)(A1)*

Therefore $a_n = Ar^n + Bnr^n$ is a solution of the given equation. *(AG)*

[6 marks]

(b) Using the result in (a), $r=3$ hence *(A1)*

$a_n = A3^n + Bn3^n$, and the two initial conditions give

$a_0 = A3^0 + B\times 0\times 3^0 = 2 \Rightarrow A = 2$, and *(M1)(A1)*

$a_1 = A3^1 + B\times 1\times 3^1 = 3 \Rightarrow B = -1$, *(A1)*

Thus the solution of the equation is

$a_n = 2\times 3^n - n3^n,\ n \ge 0$. *(A1)*

[5 marks]

Total [17 marks]

4. (i) (a) $\int_{0.4}^{1.6} \frac{\sin x}{x} \mathrm{d}x \approx \int_{0.4}^{1.6} \left(\frac{\sin 1}{x} + \ldots - (\cos 1) \times \frac{(x-1)^3}{6x} \right) \mathrm{d}x = 0.991$ ***(M1)(A1)***

[2 marks]

(b) Simpson's rule

0.4	0.6	0.8	1	1.2	1.4	1.6
0.97355	0.94107	0.89670	0.84147	0.77670	0.70389	0.62473
0.97355	4*0.94107	2*0.89670	4*0.84147	2*0.77670	4*0.70389	0.62473

(M1)

$$\frac{1.6-0.4}{18}(0.97355 + 4 \times 0.94107 + 2 \times 0.89670 + 4 \times 0.84147 + 2 \times 0.77670 + 4 \times 0.70389 + 0.62473)$$ ***(M1)(A1)***

$= 0.99272$ ***(A1)***

[4 marks]

(ii) We use the ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(n+1)x}{\mathrm{e}^{(n+1)x}}}{\dfrac{nx}{\mathrm{e}^{nx}}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x\mathrm{e}^{nx}}{nx\mathrm{e}^{(n+1)x}}\right|$$ ***(M1)(A1)***

$$= \lim_{n\to\infty}\left|\frac{(n+1)}{n\mathrm{e}^x}\right| = \lim_{n\to\infty}\frac{(n+1)}{n} \times \frac{1}{\mathrm{e}^x} = \frac{1}{\mathrm{e}^x}$$ ***(A1)***

But, for $x > 0$, $\frac{1}{\mathrm{e}^x} < 1$ and the series will converge. For $x < 0$, it will diverge. ***(R1)***

Also, for $x = 0$, the series is of the form $0 + 0 + \ldots$ which obviously converges. ***(A1)***

Therefore the series will converge for $x \geq 0$. ***(A1)***

[6 marks]

continued ...

Question 4 continued

(iii) (a) $g'(x) = 1 + x^4 + \ldots + x^{4n-4} + \ldots$ and this new series converges over any subinterval of $]-1, 1[$. ***(M1)***

This series is a geometric series with first term 1 and common ratio x^4, hence $g'(x) = \dfrac{1}{1-x^4}$. ***(R1)(M1)(AG)***

[3 marks]

(b) $\dfrac{1}{1-x^4} \equiv \dfrac{ax+b}{1+x^2} + \dfrac{c}{1-x} + \dfrac{d}{1+x}$ ***(M1)(A1)***

$1 \equiv (ax+b)(1-x^2) + c(1+x^2)(1+x) + d(1+x^2)(1-x)$

$1 \equiv (-a+c-d)x^3 + (-b+c+d)x^2 + (a+c-d)x + b+c+d$

$$\begin{cases} -a+c-d=0 \\ -b+c+d=0 \\ a+c-d=0 \\ b+c+d=1 \end{cases} \Rightarrow a=0,\, b=\frac{1}{2},\, c=\frac{1}{4},\, d=\frac{1}{4}$$

$g(x) = \displaystyle\int \frac{\mathrm{d}x}{1-x^4} = \frac{1}{2}\int \frac{\mathrm{d}x}{1+x^2} + \frac{1}{4}\int \frac{\mathrm{d}x}{1-x} + \frac{1}{4}\int \frac{\mathrm{d}x}{1+x} =$ ***(M1)(A1)***

$= \dfrac{1}{2}\arctan x + \dfrac{1}{4}\ln\dfrac{1+x}{1-x} + C$ ***(M1)(A1)***

From the given series $g(0) = 0$, and hence $C = 0$,

Therefore $g(x) = \dfrac{1}{2}\arctan x + \dfrac{1}{4}\ln\dfrac{1+x}{1-x}$. ***(A1)***

[7 marks]

Total [22 marks]

5. (a) The equation of the parabola is $y^2 = 4px$, ***(A1)***

hence the derivative function is given by $y' = \dfrac{2p}{y}$ ***(M1)(A1)***

and at point (x_0, y_0) the equation of the tangent will be

$y - y_0 = \dfrac{2p}{y_0}(x - x_0)$. ***(A1)***

The y-coordinate of T is $y = 0$, hence

$-y_0 = \dfrac{2p}{y_0}(x-x_0) \Rightarrow -y_0^2 = 2p(x-x_0) \Rightarrow -4px_0 = 2px - 2px_0 \Rightarrow x = -x_0$ ***(M1)(A1)(AG)***

[6 marks]

continued ...

Question 5 continued

(b) Equation of normal: $y - y_0 = -\dfrac{y_0}{2p}(x - x_0)$. ***(M1)***

The y-coordinate of N is $y = 0$,

hence $-y_0 = -\dfrac{y_0}{2p}(x - x_0) \Rightarrow -2py_0 = -y_0(x - x_0) \Rightarrow -2p = -x + x_0$ ***(M1)(A1)***

$\Rightarrow x = x_0 + 2p$

$x = \text{ON} = \text{OK} + \text{KN} = x_0 + \text{KN} = x_0 + 2p \Rightarrow \text{KN} = 2p$. ***(M1)(A1)***

[5 marks]

(c) Since, $\text{FT} = p + |\text{OT}| = p + x_0$ and $\text{MF} = \text{MA}$ by definition of the parabola. ***(A1)(R1)***

Also, $\text{MA} = x_0 + p$, then $\text{FM} = \text{FT}$. ***(R1)***

But triangle TMN is a right triangle, as MN is a normal and MT is a tangent to the parabola at the same point. ***(R1)***

Therefore F is the centre of the circle whose diameter is TN (angle inscribed in a semi circle). ***(R1)***

Alternate proof goes through coordinates as in (a):

Equation of normal: $y - y_0 = -\dfrac{y_0}{2p}(x - x_0)$. The y-coordinate of N is $y = 0$, hence $-y_0 = -\dfrac{y_0}{2p}(x - x_0) \Rightarrow -2py_0 = -y_0(x - x_0) \Rightarrow -2p = -x + x_0$

$\Rightarrow x = x_0 + 2p$.

But, $x = \text{ON} = \text{OF} + \text{FN} = p + \text{FN} \Rightarrow \text{FN} = x_0 + p$, and

$\text{FT} = p + |\text{OT}| = p + x_0$, and hence F is the midpoint of TN as required.

[5 marks]

(d) From (c) above, $\text{MA} = \text{MF}$, and since F is the midpoint of TN, then $\text{MA} = \text{FN}$ ***(R1)***

and they are also parallel. So, AMNF is a parallelogram and AF is parallel to MN. ***(R1)***

But since MT is perpendicular to MN, then it is perpendicular to AF. Therefore MT bisects angle AMF. ***(R1)(R1)***

Now in triangle AMC, MB is the internal bisector of the vertex angle M, MN being perpendicular to the internal bisector becomes the external bisector of the vertex angle, hence by the bisector theorem ***(R1)***

$\dfrac{\text{AB}}{\text{BC}} = \dfrac{\text{AM}}{\text{MC}}$ and $\dfrac{\text{AN}}{\text{CN}} = \dfrac{\text{AM}}{\text{MC}}$ ***(M1)***

therefore $\dfrac{\text{AB}}{\text{BC}} = \dfrac{\text{AN}}{\text{CN}}$. ***(A1)***

Therefore (ACBN) is a harmonic division. ***(AG)***

[7 marks]

Total [23 marks]