MARKSCHEME

MAY 2005

FURTHER MATHEMATICS

Standard Level

Paper 2

Paper 2 Markscheme

Instructions to Examiners

1 Method of marking

(a) All marking must be done using a **red** pen.

(b) Marks should be noted on candidates' scripts as in the markscheme:

- show the breakdown of individual marks using the abbreviations ***(M1)***, ***(A2)*** *etc*.
- write down each part mark total, indicated on the markscheme (for example, ***[3 marks]***) – it is suggested that this be written at the end of each part, and underlined;
- write down and circle the total for each question at the end of the question.

2 Abbreviations

The markscheme may make use of the following abbreviations:

M Marks awarded for **Method**

A Marks awarded for an **Answer** or for **Accuracy**

G Marks awarded for correct solutions, generally obtained from a **Graphic Display Calculator**, irrespective of working shown

R Marks awarded for clear **Reasoning**

AG **Answer Given** in the question and consequently marks are **not** awarded

3 Follow Through (ft) Marks

Errors made at any step of a solution can affect all working that follows. To limit the severity of the penalty, **follow through (ft)** marks should be awarded. The procedures for awarding these marks require that all examiners:

(i) penalise an error when it **first occurs**;

(ii) **accept the incorrect answer** as the appropriate value or quantity to be used in all subsequent working;

(iii) award ***M*** marks for a correct method, and ***A*****(ft)** marks if the subsequent working contains no further errors.

Follow through procedures may be applied repeatedly throughout the same problem.

The following illustrates a use of the **follow through** procedure:

Markscheme		Candidate's Script	Marking	
\$ 600 × 1.02	***M1***	Amount earned = \$ 600 × 1.02	√	***M1***
= \$ 612	***A1***	= \$602	×	***A0***
\$ (306 × 1.02) + (306 × 1.04)	***M1***	Amount = 301 × 1.02 + 301 × 1.04	√	***M1***
= \$ 630.36	***A1***	= \$ 620.06	√	***A1*(ft)**

Note that the candidate made an arithmetical error at line 2; the candidate used a correct method at lines 3, 4; the candidate's working at lines 3, 4 is correct.

However, if a question is transformed by an error into a **different, much simpler question** then:

(i) **fewer** marks should be awarded at the discretion of the Examiner;

(ii) marks awarded should be followed by "**(d)**" (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(iii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

4 Using the Markscheme

(a) This markscheme presents a particular way in which each question may be worked and how it should be marked. **Alternative methods** have not always been included. Thus, if an answer is wrong then the working must be carefully analysed in order that marks are awarded for a different method in a manner which is consistent with the markscheme.

In this case:

(i) a mark should be awarded followed by "**(d)**" (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(ii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

Where alternative methods for complete questions are included, they are indicated by **METHOD 1**, **METHOD 2**, *etc*. Other alternative solutions, including graphic display calculator alternative solutions are indicated by **OR**. For example:

Mean = 7906/134 ***(M1)***
= 59 ***(A1)***

OR

Mean = 59 ***(G2)***

(b) Unless the question specifies otherwise, accept **equivalent forms**. For example: $\frac{\sin\theta}{\cos\theta}$ for $\tan\theta$.

On the markscheme, these equivalent numerical or algebraic forms will generally be written in brackets after the required answer. Paper setters will indicate the required answer, by allocating full marks at that point. Further working should be ignored, even if it is incorrect. For example: if candidates are asked to factorize a quadratic expression, and they do so correctly, they are awarded full marks. If they then continue and find the roots of the corresponding equation, do not penalize, even if those roots are incorrect ie, once the correct answer is seen, ignore further working.

(c) As this is an international examination, all **alternative forms of notation** should be accepted. For example: 1.7, $1\cdot 7$, 1,7; different forms of vector notation such $\vec{u}$, $\bar{u}$, $\underline{u}$, $\tan^{-1} x$ for arctan x.

5 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

There are two types of accuracy error. Candidates should be penalized **once only IN THE PAPER** for an accuracy error **(AP)**.

Award the marks as usual then write –1**(AP)** against the answer and also on the **front** cover.

Rounding errors: only applies to final answers not to intermediate steps.

Level of accuracy: when this is not specified in the question the general rule *unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures* applies.

- If a final correct answer is incorrectly rounded, apply the **AP**.

  OR

- If the level of accuracy is not specified in the question, apply the **AP** for answers not given to 3 significant figures. (Please note that this has changed from May 2003).

Incorrect answers are wrong, and the accuracy penalty should not be applied to incorrect answers.

Examples

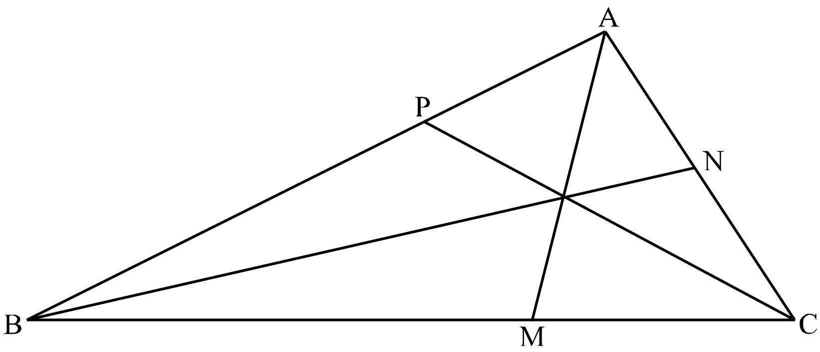
A question leads to the answer 4.6789….
- 4.68 is the correct 3 s.f. answer.
- 4.7, 4.679 are to the wrong level of accuracy : both should be penalised the first time this type of error occurs.
- 4.67 is incorrectly rounded - penalise on the first occurrence.

Note: All these "incorrect" answers may be assumed to come from 4.6789..., even if that value is not seen, but previous correct working is shown. However, 4.60 is wrong, as is 4.5, 4.8, and these should be penalised as being incorrect answers, not as examples of accuracy errors.

6 Graphic Display Calculators

Many candidates will be obtaining solutions directly from their calculators, often without showing any working. They have been advised that they must use mathematical notation, not calculator commands when explaining what they are doing. Incorrect answers without working will receive no marks. However, if there is written evidence of using a graphic display calculator correctly, method marks may be awarded. Where possible, examples will be provided to guide examiners in awarding these method marks.

1. (i)



(a) Since CP is a bisector to angle C in triangle ABC, we can use the bisector theorem

$$\frac{AP}{PB} = \frac{AC}{CB}$$ ***(M1)***

Then, using the sine rule,

$$\frac{AC}{CB} = \frac{\sin\frac{\pi}{8}}{\sin\frac{5\pi}{8}} = \frac{\sin\frac{\pi}{8}}{\cos\frac{\pi}{8}} = \tan\frac{\pi}{8}$$ ***(M1) (A1)***

(b) Triangle ACM is isosceles with AM = CM ***(A1)***

$$\frac{BM}{CM} = \frac{BM}{AM} = \tan B\hat{A}M = \tan\frac{3\pi}{8}$$ ***(M1) (A1)***

(c) We calculate the product

$$\frac{BM}{MC} \times \frac{CN}{NA} \times \frac{AP}{PB} = \tan\frac{\pi}{8} \times \tan\frac{3\pi}{8} \times 1 = \tan\frac{\pi}{8} \times \cot\frac{\pi}{8} \times 1$$ ***(M1) (A1)***

$$= 1$$ ***(A1)***

By the converse of Ceva's theorem the lines (AM), (BN) and (CP) are concurrent. ***(R1)***

(ii) (a) Let x_1 and x_2 be the coordinates of D and F, then

$$D\left(x_1, 2\sqrt{x_1}\right), \quad F\left(x_2, -2\sqrt{x_2}\right)$$ ***(M1)***

and since E is the midpoint of DF

$$\frac{x_1 + x_2}{2} = 2; \quad \frac{2\sqrt{x_1} - 2\sqrt{x_2}}{2} = 1 \Leftrightarrow x_1 + x_2 = 4; \quad \sqrt{x_1} - \sqrt{x_2} = 1$$ ***(M1)***

The equation of the line is

$$y - 1 = \frac{y_1 - y_2}{x_1 - x_2}(x-2) = \frac{2\sqrt{x_1} + 2\sqrt{x_2}}{x_1 - x_2}(x-2) = \frac{2}{\sqrt{x_1} - \sqrt{x_2}}(x-2)$$

$$\Rightarrow y - 1 = 2(x-2) \Rightarrow y = 2x - 3$$ ***(M1)(A1)***

continued ...

Question 1 (ii) continued

(b) Substituting $y = 2x + 3$ into the equation of the parabola

$y^2 = 4x \Rightarrow (2x+3)^2 = 4x \Rightarrow 4x^2 - 16x + 9 = 0$

and so $x = \dfrac{4 \pm \sqrt{7}}{2}$ ***(M1)(A1)***

Substituting into the original equation will give the coordinates of

D and F as $\left(\dfrac{4-\sqrt{7}}{2}, 1-\sqrt{7}\right), \left(\dfrac{4+\sqrt{7}}{2}, 1+\sqrt{7}\right)$ ***(A1)(A1)***

> **Note:** Alternative method:
> Substituting $x_2 = 4 - x_1$ into the second equation and squaring twice gives
> $4x_1^2 - 16x_1 + 9 = 0$
> So $x_1 = \dfrac{4 \pm \sqrt{7}}{2}$, i.e. x_1 and x_2 are conjugates.
> Substituting into the original equation will give the coordinates of D and F as $\left(\dfrac{4-\sqrt{7}}{2}, 1-\sqrt{7}\right), \left(\dfrac{4+\sqrt{7}}{2}, 1+\sqrt{7}\right)$
> The equation of the line follows, i.e. $y = 2x - 3$

(c) The distance

DF $= \sqrt{\left(\dfrac{4+\sqrt{7}}{2} - \dfrac{4-\sqrt{7}}{2}\right)^2 + \left(1+\sqrt{7} - \left(1-\sqrt{7}\right)\right)^2} = \sqrt{35}$ ***(M1)(A1)***

2. (a) $f(0) = -5, f(1) = 1$ ***(A1)***

Since f is a continuous function and $f(0) \times f(1) < 0$, ***(M1)***

by the intermediate value theorem, there must be at least one value of $c \in]0, 1[$ such that $f(c) = 0$. ***(R1)***

(b) (i) $f : I \to \mathbb{R}$, where I is an open interval, $I \subseteq \mathbb{R}$, F is continuous and differentiable on I. ***(A1)***

Given that $f(a) = f(b), a, b \in I, a \neq b$, then there exist at least one c, $a < c < b$ such that $f'(c) = 0$. ***(A1)***

(ii) If there were two zeroes on $]0, 1[$, let us call them a and b. Then by the Rolle's theorem there must be $a < c < b$ such that $f'(c) = 0$. ***(R1)***

But $f'(x) = 5x^4 + 6x^2 + 3$ has no real zeroes ***(A1)***

Therefore there is only one zero of the function f on $]0, 1[$. ***(R1)(AG)***

(c) $f(x) = x^5 + 2x^3 + 3x - 5, f'(x) = 5x^4 + 6x^2 + 3, \quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_0 = 1$	$x_0 = 0$	
$x_1 = 0.9285714286$	$x_1 = 1.666666667$	***(A1)***
	$x_2 = 1.28691536$	
$x_2 = 0.9220632235$	$x_3 = 1.037274638$	***(A1)***
	$x_4 = 0.9359415572$	
$x_3 = 0.9220143819$	$x_5 = 0.9222335721$	***(A1)***
$x_4 = 0.9220143791$	$x_6 = 0.922014434$	

$x = 0.922014$ ***(A1)***

(d) (i) $x^5 + 2x^3 + 3x - 5 = 0 \Rightarrow 2x^3 = 5 - x^5 - 3x \Rightarrow$ ***(M1)***

$x^3 = \dfrac{5 - x^5 - 3x}{2} \Rightarrow x = \sqrt[3]{\dfrac{5 - x^5 - 3x}{2}}$ ***(A1)(AG)***

(ii) $g(x) = \sqrt[3]{\dfrac{5 - x^5 - 3x}{2}} \Rightarrow |g'(1)| = 2.12 > 1$, so the iteration diverges. ***(M1)(A1)***

(e) (i) $A_S = \dfrac{2-1}{3 \times 2}\left(f(1) + 4 \times f(1.5) + f(2)\right) =$ ***(M1)(A1)***

$= \dfrac{1}{6}\left(1 + 4 \times \dfrac{443}{32} + 49\right) = \dfrac{281}{16} = 17.5625$ ***(A1)***

(ii) Exact value: $\int_1^2 f(x)\,dx = \dfrac{35}{2} = 17.5$ ***(G1)***

Percentage error: $\dfrac{\left|\dfrac{281}{16} - \dfrac{35}{2}\right|}{\dfrac{35}{2}} \times 100\,\% = 0.357\,\%$ ***(A1)***

3. (i) (a) The order of vertices in each graph is 4, 3, 3, 3, 3
so we can construct an isomorphism ϕ such that ϕ(D) = J
because they are both of order 4. ***(R1)(A1)***
A possible mapping of the rest is
ϕ(A) = F, ϕ(B) = G, ϕ(C) = H, ϕ(E) = I. ***(A1)***

(b) (i) κ_6 has $\binom{6}{2} = 15$ edges. ***(A1)***

(ii) To be a κ_4 any set of four vertices in G should contain exactly one vertex from the following three pairs of vertices:
P or Q, R or S, T or U.
Nevertheless, G must contain four vertices, therefore we need to add one more vertex.
This fourth vertex has to be one of the three vertices not chosen in the first round.
However, any of these vertices is not connected to every vertex in the already chosen set, and hence no κ_4 can be formed. ***(R2)***

(iii) There are two cases:
Disjoint, e.g. PQ and RS, but then the set {T, U, P or Q, R or S} will form κ_4.
Sharing a vertex, e.g. PQ and QR, but then the set {S, T, U, P or Q or R} will form κ_4.
Hence removal of only two edges will leave a graph which must contain κ_4. ***(M2)***
Therefore the number of edges in a maximum graph with 6 vertices is $15 - 3 = 12$ ***(A1)***

(ii) (a) If x and y have the same remainder r when divided by m, then

$$\left.\begin{aligned} x &= k_1 m + r \\ y &= k_2 m + r \end{aligned}\right\} \Rightarrow x - y = (k_1 - k_2)m$$ ***(M1)***

$(k_1 - k_2) \in \mathbb{Z} \Rightarrow x \equiv y \pmod{m}$ ***(A1)(AG)***

$x \equiv y \pmod{m} \Rightarrow x - y = km$
Suppose x and y do not have the same remainder when divided by m ***(A1)***

$$\left.\begin{aligned} x &= k_1 m + r_1 \\ y &= k_2 m + r_2 \end{aligned}\right\} \text{ with } r_1, r_2 < m$$ ***(M1)***

$\Rightarrow x - y = (k_1 - k_2)m + (r_1 - r_2) = km$
$\Rightarrow r_1 - r_2 = 0 \Rightarrow r_1 = r_2$ ***(A1)(AG)***

continued ...

Question 3 (ii) continued

(b) $\left.\begin{array}{l}2x \equiv 3 \pmod 5\\ 3x \equiv 2 \pmod 7\end{array}\right\} \Rightarrow \left.\begin{array}{l}2x-3=5k\\ 3x-2=7l\end{array}\right\} \Rightarrow \left.\begin{array}{l}6x-9=15k\\ 6x-4=14l\end{array}\right\}$ ***(M1)(A1)***

$-5 = 15k - 14l \Rightarrow k = \dfrac{14l-5}{15}$ ***(M1)(A1)***

Since all the values are positive integers by inspection we get ***(A1)***

$l = 10,\ k = 9 \Rightarrow x = 24$

OR

$\left.\begin{array}{l}2x \equiv 3 \pmod 5\\ 3x \equiv 2 \pmod 7\end{array}\right\}$ ***(M1)(A1)***

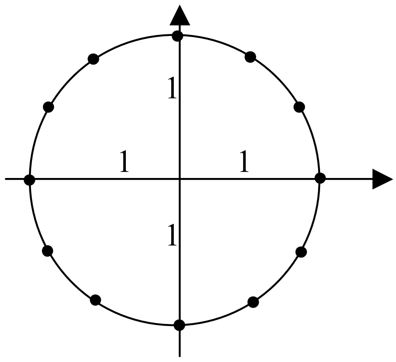
Multiply the first equation by 3 and the second by 5

$\Rightarrow \left.\begin{array}{l}x \equiv 4 \pmod 5\\ x \equiv 3 \pmod 7\end{array}\right\} \Rightarrow \left.\begin{array}{l}x = 4+5k\\ x = 3+7l\end{array}\right\} \Rightarrow$ ***(M1)(A1)***

$\left.\begin{array}{l}x = 4,\ 9,\ 13,\ 19,\ 24,\ 29\ \ldots\\ x = 3,\ 10,\ 17,\ 24,\ 31,\ 38\ \ldots\end{array}\right\} \Rightarrow x = 24$ ***(A1)***

4. (i) If $(A \cap B') \cup B = B$ ***(M1)(A1)***
then $A \cup B = B$ and $A \subset B$ ***(R1)***
If $A \subset B$ then $A \cap B' = \varnothing \Rightarrow \varnothing \cup B = B$ ***(A1)(A1)***

(ii) (a) (i) $\omega = \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}$, $\omega^2 = \cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}$, $\omega^3 = \mathrm{i}$, $\omega^4 = \cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}$
$\omega^5 = \cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}$, $\omega^6 = -1$, $\omega^7 = \cos\frac{7\pi}{6} + \mathrm{i}\sin\frac{7\pi}{6}$, $\omega^8 = \cos\frac{4\pi}{3} + \mathrm{i}\sin\frac{4\pi}{3}$
$\omega^9 = -\mathrm{i}$, $\omega^{10} = \cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}$, $\omega^{11} = \cos\frac{11\pi}{6} + \mathrm{i}\sin\frac{11\pi}{6}$, $\omega^{12} = 1$ ***(A3)***



(ii) The numbers form a regular dodecagon in the complex plane inscribed in the unit circle. ***(A2)***

(iii) The possible generators are $\{\omega, \omega^5, \omega^7, \omega^{11}\}$ ***(A2)***

(b) All the possible proper subgroups are:
$\{1, -1\}$, $\{1, \omega^4, \omega^8\}$, $\{1, \mathrm{i}, -1, -\mathrm{i}\}$, $\{1, \omega^2, \omega^4, -1, \omega^8, \omega^{10}\}$ ***(A4)***

(c)

×	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

(A1)

×	1	i	–1	–i
1	1	i	–1	–i
i	i	–1	–i	1
–1	–1	–i	1	i
–i	–i	1	i	–1

We can observe that both groups are cyclic groups of order 4, therefore it is possible to construct an isomorphism. It is possible to construct two different isomorphisms. ***(R1)***
$1 \to 1$, $4 \to -1$ and then we have two possibilities
$2 \to \mathrm{i}$ or $-\mathrm{i}$, while $3 \to -\mathrm{i}$ or i. ***(M1)(A1)***

5. (a) $\bar{x}_1 = 2.26, s_1 = 1.60$ ***(A1)(A1)***

(b) $X \sim \text{Po}(\mu = 2.26) \Rightarrow E_i = 100 \times P(X = i), i = 0, 1...$ ***(M1)***

x_i	0	1	2	3	4	5 or more
f_o	21	11	24	18	17	9
f_e	10.4	23.6	26.6	20.1	11.3	8

(A3)

H_0: The distribution can be modelled by a Poisson distribution.
H_1: The distribution can not be modelled by a Poisson distribution. ***(A1)***

$$\chi^2_{calc} = \sum_{i=0}^{5} \frac{(f_e - f_o)^2}{f_e} = 21.0$$ ***(M1)(A1)***

$\nu = 6 - 2 = 4$ degrees of freedom. ***(A1)***

$\chi^2 = 9.488$ ***(A1)***

Since 21 > 9.49 we reject $H_{0\,,}$ i.e. the distribution cannot be modelled by a Poisson distribution. ***(R1)***

(c) H_0: There is no difference between the means.
H_1: There is a difference between means. ***(A1)***
We use a two-tailed test at the 5 % level of significance. ***(M1)***

EITHER

$$z = \frac{2.26 - 2}{\sqrt{\frac{2.5524}{100} + \frac{0.9^2}{80}}} = 1.38$$ ***(M1)(A1)***

Since 1.38 < 1.96 we do not have enough evidence to reject H_0 and conclude that there is no sufficient evidence to show that there is a difference between two mean values of fouls per game. ***(R1)***

OR

$$t = \frac{2.26 - 2}{1.34\sqrt{\frac{1}{100} + \frac{1}{80}}} = 1.29$$, 178 degrees of freedom ***(M1)(A1)***

Since 1.29 < 1.97 we do not have enough evidence to reject H_0 and conclude that there is no sufficient evidence to show that there is a difference between two mean values of fouls per game. ***(R1)***

(d) H_0: There is no difference in the fouls per game.
H_1: The first player makes more fouls per game. ***(A1)***
We use one-tailed test at the 10 % level of significance. ***(M1)***

EITHER

$z = 1.377$

Since 1.38 >1.282 we reject H_0 and conclude that there is evidence that first player makes more fouls per game. ***(R1)***

OR

$t = 1.29$, 178 degrees of freedom

Since 1.29 >1.286 we reject H_0 and conclude that there is evidence that first player makes more fouls per game. ***(R1)***